# Math 241, Spring 2026 — Homework 4

Tim Chumley

Due February 26 at 5:00 pm

**Instructions.** This problem set covers material from Week 4 of class. The problems synthesize a number of questions from our textbook, *A First Course in Chaotic Dynamical Systems*, by Robert L. Devaney.

**Problem 1.** Let $F_\lambda : \mathbb{R} \to \mathbb{R}$ be given by $F_\lambda(x) = \lambda x(1-x)$ where $\lambda > 0$ is a given constant.

a. It is straightforward to see that 0 is always a fixed point of $F_\lambda$. Find the values of $\lambda$ for which 0 is attracting. Do the same for when 0 is neutral and repelling.

b. Find the values of $\lambda$ for which this system has a second fixed point $p$ which is positive. Please write your answer as an interval $I$. Please also give an expression for $p$ in terms of $\lambda$.

c. How is the appearance of a positive fixed point related to 0 being attracting/repelling/neutral?

d. Suppose $\lambda \in I$. Find the values of $\lambda$ for which $p$ is attracting. Do the same for when it is neutral and repelling.

e. Consider the equation $F_\lambda^2(x) = x$. Using algebra, we can find that, besides 0 and $p$, this equation has solutions given by

$$q_\pm = \frac{\lambda+1}{2\lambda} \pm \frac{1}{2\lambda}\sqrt{(\lambda-1)^2-4}.$$

For which values of $\lambda$ does the system have a 2-cycle?

f. How is the appearance of a 2-cycle related to $p$ being attracting/repelling/neutral?

*Remark* 1. It can be helpful for you to have Desmos open as you do this problem. Try plotting the graphs of $y = F_\lambda(x)$, $y = F_\lambda^2(x)$, and $y = x$ and watch how periodic points and 2-cycles appear as you slide the value of $\lambda$ around. Pay attention to the slopes at the fixed points as you do this.

**Problem 2.** Suppose that $F : \mathbb{R} \to \mathbb{R}$ is an odd function. This means $F(-x) = -F(x)$ for all $x \in \mathbb{R}$. Suppose further that $x_0 \neq 0$ satisfies the equation $F(x) = -x$. Prove that $x_0$ is periodic with prime period 2.

*Remark* 2. Note that this gives us an algebraic method for finding 2-cycles when $F$ is an odd function. We just solve the equation $F(x) = -x$.

**Problem 3.** Let $G_\lambda(x) = x^3 + \lambda x$ where $\lambda \in \mathbb{R}$ is a given constant. Note that $G_\lambda$ is an odd function for each $\lambda \in \mathbb{R}$.

a. Use the technique of Problem 2 to find values of $\lambda$ for which $G_\lambda$ has a 2-cycle. Please write your answer as an interval $I$. Please also express the elements $q_1$ and $q_2$ of the 2-cycle in terms of $\lambda$.

b. Compute $(G^2)'(q_1)$. Your answer will be a formula in terms of $\lambda$. Please simplify it as much as possible. Note that the expression you get will be a non-negative number for all $\lambda \in I$.

c. Find all values of $\lambda \in I$ for which the 2-cycle is attracting. Do the same for neutral and repelling.

d. Use Desmos to plot the graphs of $y = G_\lambda(x)$, $y = G_\lambda^2(x)$, and $y = x$ on the same axes. Do this three times: (1) for a value of $\lambda$ that makes the 2-cycle attracting, (2) for a value of $\lambda$ that makes the 2-cycle neutral, and (3) for a value of $\lambda$ that makes the 2-cycle repelling.

e. Suppose $\lambda$ is a value for which $q_1, q_2$ give a repelling 2-cycle. Use your previous answer to state how many prime period 2 points the system has.

f. How is the appearance of additional 2-cycles related to $q_1, q_2$ being an attracting/repelling/neutral 2-cycle?