Math 241, Spring 2026 — Homework 5

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Due March 5 at 5:00 pm

Instructions. This problem set covers material from Week 5 of class. The chapter and exercise numbers refer our textbook, *A First Course in Chaotic Dynamical Systems*, by Robert L. Devaney.

Problem 1. Please do the following exercises from chapter 6: 1efghi, 9, 11

Remark 1. In Exercise 6.1g, the bifurcation value should be $c = 2/(3\sqrt{3})$.

Remark 2. In Exercise 6.1, for all parts, you may justify your answers graphically and avoid doing algebra/calculus. For example, if you believe a saddle-node bifurcation occurs at λ_0, you may justify this by showing a series of three Desmos plots: one which shows a λ value where there are no fixed points in an open interval I, one which shows a λ value where there is 1 neutral fixed point in I, and one which shows a λ value where there are 2 fixed points in I, one of which is repelling and one of which is attracting (making sure to point out which is which). If you believe the bifurcation is neither a saddle-node nor a period-doubling bifurcation, make sure to explain why.

Remark 3. In Exercises 6.9 and 6.11, you should use the work you did in Homework 4, Problem 1. You may use Desmos to plot your bifurcation diagrams.

Problem 2. The 1-parameter family $F_\lambda(x) = x^2 + x - \lambda$ undergoes a saddle-node bifurcation at $\lambda_0 = 0$.

a. When $\lambda > 0$, the system has two fixed points, p_- and p_+. Give formulas for these in terms of λ.

b. Find a value $\epsilon > 0$ such that for all $\lambda \in (\lambda_0, \lambda_0 + \epsilon)$ one of the two fixed points you found in the previous part is repelling and the other is attracting.

Problem 3. The 1-parameter family $G_\lambda(x) = x^3 + \lambda x$ undergoes a period-doubling bifurcation. This problem asks you to show this by formally checking the definition. Please use your work on Homework 4, Problem 3, to answer (and justify your answers to) the following questions.

a. What is the bifurcation value λ_0? Just an answer without justification is ok here.

b. Find an open interval $I \subseteq \mathbb{R}$ and a real number $\epsilon > 0$ so that

   1. when $\lambda \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon)$, there is a unique fixed point p_λ for G_λ in I
   2. when $\lambda \in [\lambda_0, \lambda_0 + \epsilon)$, G_λ has no 2-cycles in I and p_λ is attracting
   3. when $\lambda \in (\lambda_0 - \epsilon, \lambda)$, G_λ has a unique 2-cycle $q^1_\lambda, q^2_\lambda \in I$ which is attracting and p_λ is repelling
   4. $\lim_{\lambda \to \lambda_0} q^1_\lambda = p_{\lambda_0}$ and $\lim_{\lambda \to \lambda_0} q^2_\lambda = p_{\lambda_0}$