# Math 241, Spring 2026 — Homework 8

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Due April 9 at 5:00 pm

**Instructions.** This problem set covers material from Week 10 of class. Any chapter and exercise numbers refer our textbook, *A First Course in Chaotic Dynamical Systems*, by Robert L. Devaney.

**Problem 1.** Find the real numbers with the following binary expansions.

a. $0.1100\overline{1}$

b. $0.\overline{1001}$

c. $0.1\overline{101}$

**Problem 2.** Prove that if the binary expansion of $x_0$ is $0.a_1a_2a_3\ldots$ then the binary expansion of $D(x_0)$ is $0.a_2a_3\ldots$. What can you conclude about the binary expansion of $D^n(x_0)$ for a given integer $n$? Note that this is just asking you to write out your solutions to Problem 2 and 3 from the worksheet on March 30.

**Problem 3.** A terminating binary expansion is one that ends with repeating 0's. Find terminating binary expansions of the following rational numbers.

a. $1/2^3$

b. $3/2^4$

c. $5/2^5$

d. $7/2^5$

**Problem 4.** A dyadic rational number is a rational number of the form $a/2^n$ for some integers $a, n$. Make a conjecture about the binary expansion of a dyadic rational number based on your work in Problem 3. Then use your conjecture and Problem 2 to explain why dyadic rationals are eventually fixed points under the doubling map.

**Problem 5.** Find the binary expansions of all prime period 3 points of the doubling map and group them together into their respective 3-cycles.

**Problem 6.** Let $x, y \in [0, 1]$ have binary expansions $0.x_1x_2x_3\ldots$ and $0.y_1y_2y_3\ldots$ respectively.

a. Suppose that $x_1 = y_1, x_2 = y_2, \ldots, x_{10} = y_{10}$. Give an upper bound on the value of $|x - y|$.

b. Suppose that $|x - y| < 1/100$. Find the largest value of $n$ for which you can guarantee that $x_1 = y_1, \ldots, x_n = y_n$. Update on April 12: as some people have noted, there is no largest $n$ that we can guarantee agreement in the first $n$ terms. For example, if $x = .50001$ and $y = 0.49999$ then $|x - y| < 1/100$ but the first digit of their respective binary expansions will be different (the binary expansion of $x$ will start with 1 and that of $y$ will start with 0). The problem can be done if instead we have that $x, y$ are elements of the space $\Sigma$ of binary sequences and we suppose that $d(x, y) < 1/100$ where $d$ is the distance function on $\Sigma$ that we have previously defined. Many of you did this and I gave credit for that. One lesson to learn here is that measuring the distance between two binary sequences using our distance function $d$ is qualitatively different from measuring their distance by viewing them as binary expansions for real numbers.

**Problem 7.** Consider the tent map

$$T(x) = \begin{cases} 3x & x \le 1/2 \\ 3 - 3x & x > 1/2 \end{cases}$$

and its corresponding itinerary map previously defined in the worksheet from April 1. Find periodic points of $T$ whose itineraries are given below. Note that you may assume that there is a 1-1 correspondence between itineraries and elements of the Cantor Middle-Thirds set. This will help you reduce the amount of work to do.

a. $(\overline{101})$

b. $(\overline{011})$

c. $(\overline{110})$

d. $(\overline{001})$

e. $(\overline{010})$

f. $(\overline{100})$