Problem 1. Consider the map $F(x) = x^3 - 2x^2 + x$.

a. Find all fixed points of F and classify each as attracting, repelling, neutral and weakly attracting, neutral and weakly repelling, or neutral and neither weakly attracting nor weakly repelling.

b. Sketch a cobweb diagram showing a few steps of orbits that start near the fixed points you found. (On an exam version of this problem, I would give you a plot with $y = F(x)$ and $y = x$.)

c. Sketch a phase portrait summarizing the behavior of orbits.

d. Express the set $\{x \in \mathbb{R} : \lim_{n\to\infty} F(x) = \pm\infty\}$ as an interval or union of intervals.

(a) $F(x) = x$ is $x^3 - 2x^2 = 0$

$\Rightarrow x^2(x-2) = 0$

which is solved by $x_1 = 0$, $x_2 = 2$.

Since $F'(x) = 3x^2 - 4x + 1$,

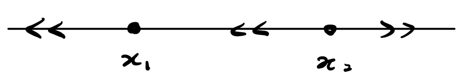
$|F'(x_1)| = 1$ and $|F'(x_2)| = 5$.

Therefore x_1 is neutral and x_2 is repelling.

A cobweb diagram shows $\lim_{n\to\infty} F^n(x) = -\infty$ when $x < x_1$

and $\lim_{n\to\infty} F^n(x) = x_1$ when $x > x_1$ and x near x_1.

So x_1 is neither weakly attracting nor weakly repelling.

(c) 

(d) $(-\infty, x_1) \cup (x_2, \infty)$.

Problem 2. Consider the 1-parameter family $F_c(x) = x^3 - 2x^2 + cx$.

a. The family undergoes bifurcations at $c = -1, 1, 2$. By looking at Desmos plots, identify each as a saddle-node bifurcation, period-doubling bifurcation, or neither. (On an exam version of this problem, I would show you plots for these and nearby parameter values.)

b. You should find one of the bifurcations is a saddle-node bifurcation. Find an interval I so that for values of c near the bifurcation value, the system goes from having 0 to 1 to 2 fixed points in I.

c. The equation $F_c(x) = x$ has solution $p_1 = 0$. Find the two others; call them p_2, p_3.

d. Find the values of c where p_1 is repelling/attracting/neutral.

e. Sketch the bifurcation diagram for the fixed points using the expressions for p_1, p_2, p_3 you found above. Label each of the three curves in your diagram with the label p_1, p_2, or p_3. No need to use dotted/solid lines to indicate repelling/attracting behavior .

f. What equation would you need to solve in order find prime period 2 points of F_c? Based on your previous work in this problem, how many real-number solutions would this equation have when $c = -1$? What about $c = -1.5$?

(a) $c=-1$ is period-doubling

$c=1$ is neither

$c=2$ is saddle node

(b) We can let $I=(0,\infty)$ or another interval that contains the positive fixed points when c is near 2 but does not contain the 0 fixed point.

(c) $x^3 - 2x^2 + cx = x \Rightarrow x^3 - 2x^2 + (c-1)x = 0$

$$\Rightarrow x(x^2 - 2x + c - 1) = 0$$

which is solved by $p_1 = 0$ and

$$p_2 = \frac{2 - \sqrt{4-4(c-1)}}{2}, \quad p_3 = \frac{2+\sqrt{4-4(c-1)}}{2}$$

$$= 1 - \sqrt{2-c} \qquad = 1 + \sqrt{2-c}$$

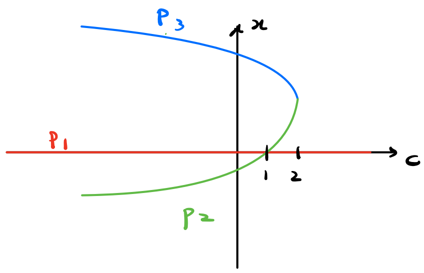
(d) $F_c'(x) = 3x^2 - 4x + c$ and so $|F_c'(p_1)| = |c|$.

There p_1 is repelling when $c \in (-\infty,-1) \cup (1,\infty)$,

p_1 is attracting when $c \in (-1,1)$

and p_1 is neutral when $c=-1$ or $c=1$.

(e)



(f) $F_c^2(x) = x$ which is a degree 4 polynomial equation

When $c = -1$, there are 3 solutions (p_1, p_2, and p_3)

and when $c = -1.5$ there are 5 solutions (p_1, p_2, p_3 and 2 prime period 2 points which can be seen by plotting $F_c(x)$ and $F_c^2(x)$)

Problem 3. Suppose that p is a neutral fixed point of F such that $F'(p) = -1$.

a. Show that p is a fixed point of F^2.

b. Show that p is a neutral fixed point of F^2.

c. The chain rule tells us $(F^2)'(x) = F'(F(x))F'(x)$. Use the product rule to find $(F^2)''(x)$.

d. Compute $(F^2)''(p)$.

(a) $F^2(p) = F(F(p))$

$= F(p)$ (since p being a fixed point means $F(p)=p$)

$= p$ (same reason as above).

(b) $(F^2)'(p) = F'(F(p))\,F'(p)$

$= F'(p)\,F'(p)$

$= (-1)(-1)$

$= 1$

(c) $(F^2)''(x) = F'(F(x)) \cdot \frac{d}{dx}\left(F'(x)\right)$

$+ \frac{d}{dx}\left(F'(F(x))\right) \cdot F'(x)$ (product rule)

$= F'(F(x)) \cdot F''(x)$

$+ F''(F(x))F'(x) \cdot F'(x)$ (chain rule)

$= F'(F(x))F''(x) + F''(F(x))\left(F'(x)\right)^2$

(d) $(F^2)''(p) = F'(F(p))F''(p) + F''(F(p))\left(F'(p)\right)^2$

$= F'(p)F''(p) + F''(p)\left(F'(p)\right)^2$

$= -F''(p) + F''(p)(-1)^2$

$= 0$

Problem 4. State whether each of the following is true or false.

a. If $F(x_0) = x_0$ then $F^3(x_0) = x_0$.

b. If $F(x)$ is a degree 2 polynomial with 2 fixed points, then it is possible for it to have 4 prime period 2 points.

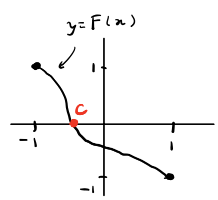
c. If $F : [-1, 1] \to [-1, 1]$ is a continuous function such that $F(-1) = 1$ and $F(1) = -1$, then $F(0) = 0$ and therefore 0 is a fixed point of F.

d. If x_0 is a neutral fixed point such that $F'(x_0) = 1$ and $F''(x_0) = 0$, then there exists an open interval I containing x_0 such that $\lim_{n\to\infty} F^n(x) = x_0$ for all $x \in I$.

(a) True, $F(x_0) = x_0$ implies $F^n(x_0) = x_0$ for all $n \geq 1$.

(b) False. When $F(x)$ is a degree 2 polynomial, $F^2(x)$ is a degree 4 polynomial and the equation $F^2(x) = x$ has at most 4 real number solutions. However, 2 of them must be fixed points. So at most 2 prime period 2 points exist.

(c) False.



The intermediate value theorem tells us there is a value c between -1 and 1 where $F(c) = 0$ but it doesn't guarantee $c = 0$.

(d) False. If $F'''(x_0) > 0$, then x_0 will be weakly repelling which prevents the desired behavior.