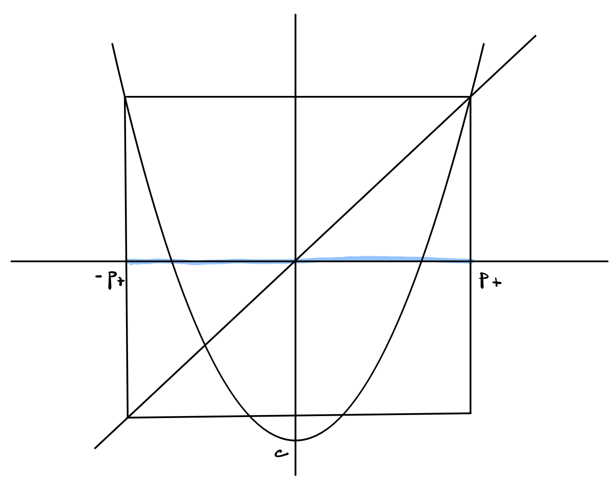
Chapter 7 $Q_c(x)$ when $c < -2$

Today we'll investigate the bounded orbits of $Q_c(x) = x^2 + c$ when $c < -2$.

Let p_+ denote the positive fixed point of Q_c and let $I = [-p_+, p_+]$



$c < -2$

Note

1. If $x_0 \notin I$, $\lim_{n \to \infty} Q_c^n(x_0) = \infty$

2. When $c < -2$, there are some $x_0 \in I$ such that $Q_c(x_0) \notin I$.

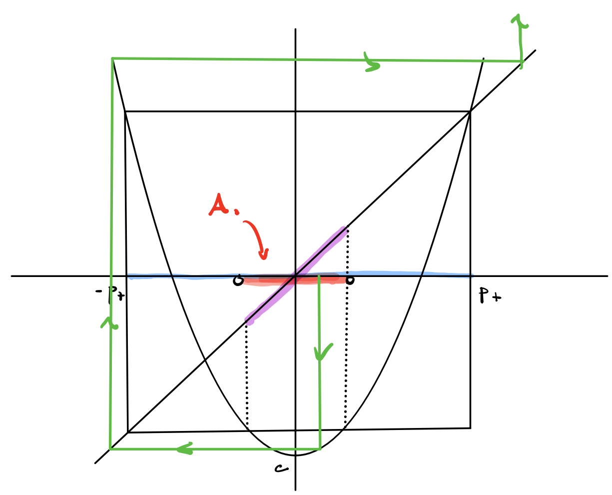
Question: Are there any $x_0 \in I$ such that $Q_c^n(x_0) \in I$ for all $n \geq 1$?

Let

$$\Lambda_1 = \{x_0 \in I : Q_c(x_0) \in I\},$$

$$\Lambda_2 = \{x_0 \in I : Q_c^2(x_0) \in I\}, \ldots$$

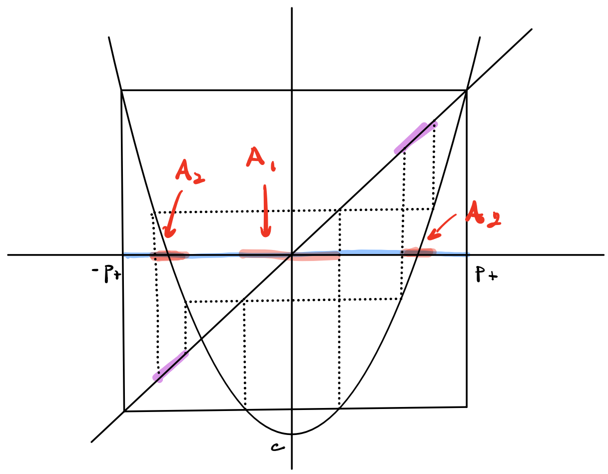
and let $\Lambda = \bigcap_{n=1}^{\infty} \Lambda_n$ be the set of seeds whose orbits stay in I. Our question is what is in Λ?



A_1 is an open interval of seeds x_0 such that $Q_c(x_0) \notin I$.

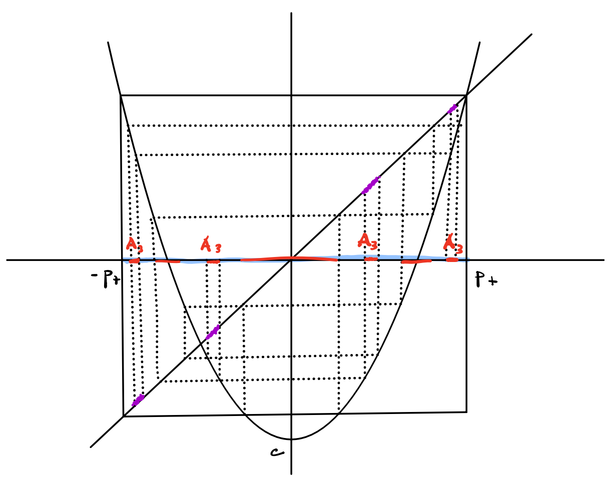
$$\Lambda_1 = I - A_1$$

set minus (alternate notation: $I \setminus A_1$)



Λ_2 is 2 open intervals in Λ_1 of initial seeds x_0 such that $Q_c^2(x_0) \notin I$.

$$\Lambda_2 = I - (A_1 \cup A_2) \subseteq \Lambda_1$$



A_3 consists of 4 open intervals in Λ_2 such that $Q_c^3(x_0) \notin I$.

$$\Lambda_3 = I - (A_1 \cup A_2 \cup A_3) \subseteq \Lambda_2 \subseteq \Lambda_1$$

Λ will consist of what remains after we repeat this process of removing intervals from I.

Notes ① Λ is not empty! For example the endpoints of $A_1, A_2, A_3, \ldots$ are never removed in this process. In fact, Λ has an uncountable number of elements.

② Λ is an example of what is called a Cantor set