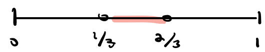
§ 7.3 Cantor Middle-Thirds Set

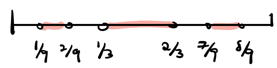
Reminder Last time we constructed the Cantor Middle-Thirds Set through an iterative process:

Step 1 :



remove $(1/3, 2/3)$ from $[0, 1]$

Step 2 :



remove the middle-thirds $(\frac{1}{9}, \frac{2}{9}) \cup (\frac{7}{9}, \frac{8}{9})$ from the 2 intervals that remain

Step 3 :

remove the middle-thirds from the 4 intervals that remain

Repeating this ad infinitum, the Cantor Middle-Thirds set Γ is the set that remains after this removal process.

Reminder A geometric series is an infinite series of the form $\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \ldots$.

When $|r|<1$, the series converges to $\frac{a}{1-r}$.

Def The ternary expansion of a real number $x \in [0,1]$ is a sequence $x_1, x_2, \ldots$ whose elements are from $\{0,1,2\}$ such that

$$x = \sum_{n=1}^{\infty} \frac{x_n}{3^n}$$

We write this as $x = 0.x_1 x_2 x_3 \ldots$.

Example What real number is represented by the ternary expansion $0.\overline{02} = 0.020202\ldots$?

$$\frac{0}{3} + \frac{2}{3^2} + \frac{0}{3^3} + \frac{2}{3^4} + \cdots$$

$$= \frac{2}{3^2} + \frac{2}{3^4} + \frac{2}{3^6} + \cdots$$

geometric series, $a = 2/9$, $r = 1/9$

$$= \frac{2/9}{1 - 1/9}$$

$$= 1/4.$$

Example What is a ternary expansion of $x = \frac{5}{8}$?

We must find $x_1, x_2, x_3, \ldots \in \{0,1,2\}$ so that

$$\frac{5}{8} = \frac{x_1}{3} + \frac{x_2}{3^2} + \frac{x_3}{3^3} + \cdots$$

Multiply both sides by 3:

$$\underbrace{\frac{15}{8}}_{1 + 7/8} = \underbrace{x_1}_{0,\,1,\text{ or }2} + \underbrace{\frac{x_2}{3} + \frac{x_3}{3^2} + \cdots}_{\leq 1}$$

To make both sides equal we need $x_1 = 1$, and

$$\frac{7}{8} = \frac{x_2}{3} + \frac{x_3}{3^2} + \frac{x_4}{3^3} + \cdots$$

Multiply both sides by 3:

$$\underbrace{\frac{21}{8}}_{2+\frac{5}{8}} = \underbrace{x_2}_{0,\,1,\text{ or } 2} + \underbrace{\frac{x_3}{3} + \frac{x_4}{3^2} + \cdots}_{\leq 1}$$

Therefore $x_2 = 2$ and

$$\frac{5}{8} = \frac{x_3}{3} + \frac{x_4}{3^2} + \frac{x_5}{3^3} + \cdots$$

Repeating the process gives that

$$x = 0.\overline{12} = 0.121212\ldots$$

Notes ① If $x \in [0, 1/3)$, its ternary expansion begins with $x_1 = 0$.

Likewise if $x \in (\frac{1}{3}, \frac{2}{3})$, its ternary expansion begins with $x_1 = 1$ and if $x \in (\frac{2}{3}, 1]$, its ternary expansion begins with $x_1 = 2$.

② Some numbers do not have a unique ternary expansion. For example, with $x = \frac{1}{3}$

$$\frac{1}{3} = \frac{1}{3} + \frac{0}{3^2} + \frac{0}{3^3} + \cdots$$

$$= \frac{0}{3} + \frac{2}{3^2} + \frac{2}{3^3} + \frac{2}{3^4} + \cdots$$

So $0.1000\ldots$ and $0.0222\ldots$ are both ternary expansions of $1/3$.

③ However, every element of Γ has a unique ternary expansion of only 0's and 2's.

Def A set X is countable if there exists an invertible function $f: \mathbb{N} \to X$

An intuitive definition A set X is countable if there is a sequence of elements of X that exhausts it, ie there exists $a_1, a_2, a_3, \ldots$ such that $X = \{a_1, a_2, \ldots\}$.

Theorem Γ is not countable.

Proof Suppose Γ is countable. Then there exists an exhaustive sequence of elements: $\Gamma = \{a_1, a_2, a_3, \ldots\}$. Each element of this sequence has a unique ternary expansion of 0's and 2's:

$$a_1 = 0.\,x_1^1 x_2^1 x_3^1 \ldots$$

$$a_2 = 0.\,x_1^2 x_2^2 x_3^2 \ldots$$

$$a_3 = 0.\,x_1^3 x_2^3 x_3^3 \ldots$$

$$\vdots$$

Let $x \in \Gamma$ be a real number with ternary expansion $x = 0.d_1 d_2 d_3 \ldots$ defined as follows:

$$d_i = \begin{cases} 0 & \text{if } x_i^i = 2 \\ 2 & \text{if } x_i^i = 0 \end{cases}$$

Notice $x \neq a_1$ since $d_1 \neq a_1^1$, $x \neq a_2$ since $d_2 \neq a_2^2$, ... So $x \neq a_n$ for all $n \geq 1$. However this contradicts that $a_1, a_2, a_3, \ldots$ was exhaustive.