Chapter 9 Symbolic Dynamics (first through a concrete example, the doubling map.)

Def Let $x \in [0,1]$. A binary expansion or binary representation of x is an infinite sequence $x_1, x_2, x_3, \ldots$ of 0's and 1's such that

$$x = \sum_{n=1}^{\infty} \frac{a_n}{2^n} = \frac{a_1}{2} + \frac{a_2}{2^2} + \frac{a_3}{2^3} + \cdots$$

We write $x = 0.a_1 a_2 a_3 \ldots$.

Example Suppose $x = 0.\overline{01} = 0.010101\ldots$ Find x.

$$0.\overline{01} = \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \cdots$$

$$= \frac{1/2^2}{1 - 1/2^2} = \frac{1/4}{1 - 1/4} = \frac{1}{3}.$$

Example Find the binary expansion of $2/3$.

$$\frac{2}{3} = \frac{a_1}{2} + \frac{a_2}{2^2} + \frac{a_3}{2^3} + \cdots$$

Multiply both sides by 2:

$$1 + \frac{1}{3} = \underbrace{a_1}_{0 \text{ or } 1} + \underbrace{\frac{a_2}{2} + \frac{a_3}{2^2} + \cdots}_{\leq \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots = \frac{1/2}{1-1/2} = 1}$$

So $a_1 = 1$ and

$$\frac{1}{3} = \frac{a_2}{2} + \frac{a_3}{2^2} + \cdots$$

We already know $\frac{1}{3} = .\overline{01}$. Therefore

$$\frac{2}{3} = 0.101010\ldots = 0.\overline{10}$$

The goal of today's worksheet will be to see how the doubling map works when we use binary expansions instead of base 10 fractions.

We'll see it has a nice structure which makes it easy to identify periodic points.