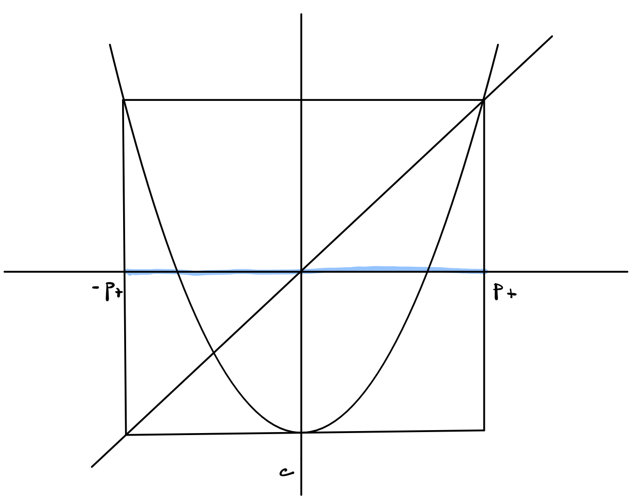
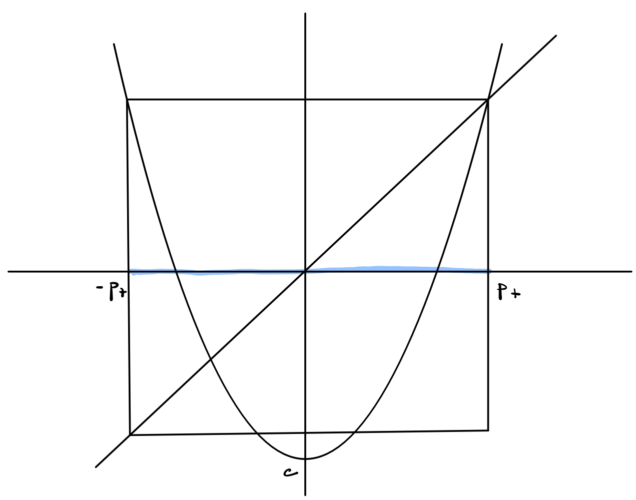
Chapter 7 The quadratic family when $c < -2$.

In this chapter we'll discuss the dynamics of $Q_c(x) = x^2 + c$ when $c < -2$.

We want to think about the behavior of initial seeds and which ones go to ∞.



$c = -2$



$c < -2$

Recall $p_+ = \frac{1+\sqrt{1-4c}}{2}$ is the right (repelling) fixed point of Q_c.

Let $I = [-p_+, p_+]$. Two observations

① If $x_0 \notin I$, then $Q_c^n(x_0) \to \infty$ as $n \to \infty$.

② When $c = -2$ and $x_0 \in I$,

$Q_{-2}^n(x_0) \in I$ for all n.

But when $c < -2$, there are some $x_0 \in I$ where $Q_c(x_0) \notin I$.

<u>Question</u> Suppose $c < -2$ and $x_0 \in I$ is chosen so that $F(x_0) \notin I$. What can you say about $F^n(x_0)$ as $n \to \infty$?

Let $\Lambda = \{ x_0 \in I : Q_c^n(x_0) \in I \text{ for all } n \}$

Question: Can we visualize the complement of Λ in I? That is, what are the points in $I \cap \Lambda^c$? Our aim is to understand Λ by understanding its complement.

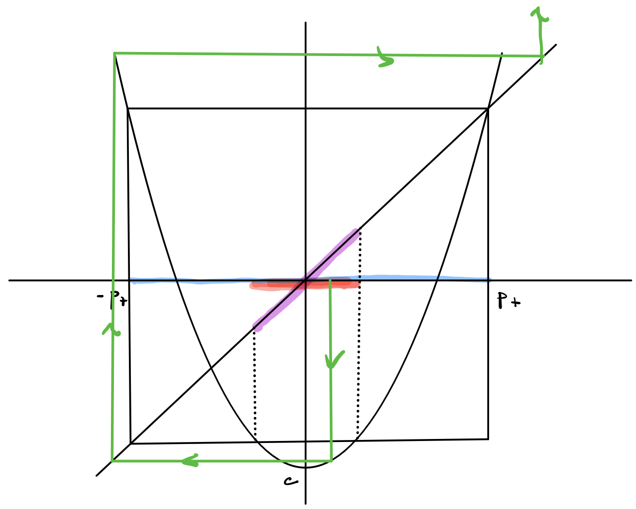
Exercise: Find $A_1 = \{ x_0 \in I : Q_c(x_0) \notin I \}$

$A_2 = \{ x_0 \in I : Q_c^2(x_0) \notin I \}$

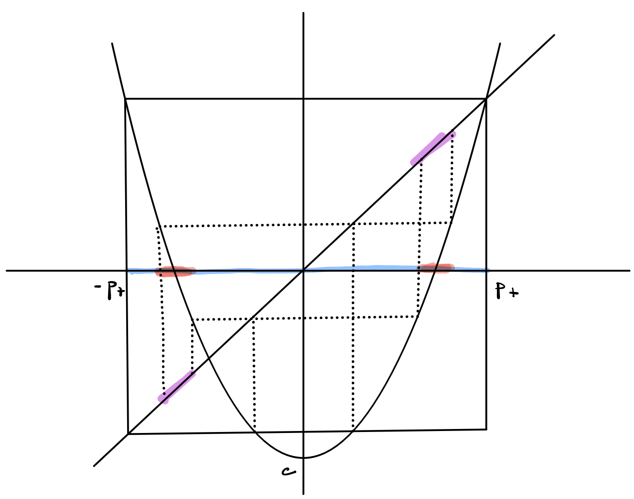
$A_3 = \{ x_0 \in I : Q_c^3(x_0) \notin I \}$

graphically. What can we say about the orbits of initial seeds from these sets?

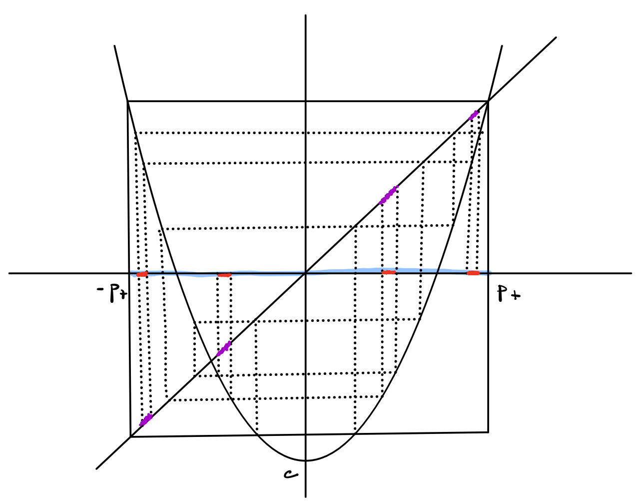
(Do this in groups)



A_1 is the set marked in red. Initial seeds here leave I in one iteration



A_2 is the set of initial seeds that leave I after two iterations. If $x_0 \in A_2$ then $Q_c^2(x_0) \notin I$. There are two intervals that comprise A_2 (marked in red)



A_3 consists of 4 intervals (marked in red) of initial seeds that result in $Q_c^3(x_0) \notin I$.

Thus Λ gets made by removing an infinite collection of sets $A_1, A_2, A_3, \ldots$ from I.

Is anything left in Λ after we remove $A_1, A_2, \ldots$?

Claim 1 The set Λ is non-empty

Proof. Even though we remove more and more intervals from I, there are some initial seeds x_0 such that $Q_c^n(x_0) \in I$ for all n. For example fixed points $p_\pm$ or the endpoints of A_n

Claim 2 The set Λ contains no intervals.

Proof A bit technical, maybe not so informative but I'll write it below.

Proof We're going to only consider the case when $c < \frac{-5+2\sqrt{5}}{4}$ because it's easier since for this range of c, $|Q_c'(x)| > 1$ for all $x \in I \cap A^c$. That is, there exists $\mu > 1$, such that $|Q_c'(x)| > \mu > 1$ for all $x \in I \cap A^c$

Suppose $J \subseteq \Lambda$ is an interval with length l.

Notice for any $x, y \in J$,

$$|Q_c(x) - Q_c(y)| = |Q_c'(z)|\,|x-y| \quad \text{for some } z \text{ between } x, y$$

$$> \mu |x-y|$$

Notice $Q_c(J) = \{Q_c(x) : x \in J\}$ is an interval too, and the above shows us $Q_c(J)$ has length greater than μl. But $Q_c^2(J)$ is an interval too and

$|Q_c^2(x) - Q_c^2(y)| > \mu^2 |x-y|$ by

applying the mean value theorem twice

and so $Q_c^2(J)$ has length $\mu^2 l$.

In general, $Q_c^n(J)$ is an interval of

length at least $\mu^n l$, but this tends

to ∞ as $n \to \infty$. However $Q_c^n(J) \subseteq I$

can't exceed the finite length of I.

So there can't be an interval $J \subseteq \Lambda$.