9.1 The itinerary map

When studying a dynamical system we can encode orbits by partitioning the phase space into a finite number of sets and keeping a record of which partition element is visited at each step.

Example Consider $T(x) = \begin{cases} 2x & 0 \leq x \leq \frac{1}{2} \\ 2-2x & \frac{1}{2} \leq x \leq 1 \end{cases}$

Given $x_0 \in [0, 1]$, we know $T^n(x_0) \in [0, 1]$ for all $n \geq 1$. We can define a sequence

$$S(x_0) = (s_0, s_1, s_2, s_3, \dots)$$

corresponding to the orbit of x_0 as follows:

$$s_n = \begin{cases} 0 & \text{if } T^n(x_0) \in [0, \frac{1}{2}) \\ 1 & \text{if } T^n(x_0) \in (\frac{1}{2}, 1] \end{cases}$$

The binary sequence $S(x_0)$ is called the itinerary of x_0.

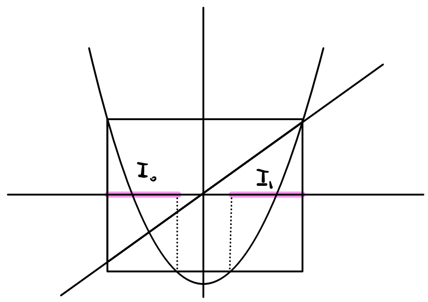
Then the orbit of 0.9 is

$$0.9,\ 0.2,\ 0.4,\ 0.8,\ 0.4,\ 0.8,\ 0.4,\ \ldots$$

The itinerary of 0.9 is

$$S(0.9) = (1, 0, 0, 1, 0, 1, 0, 1, \ldots)$$

<u>Example</u> Consider $Q_c(x) = x^2 + c$ where $c < -2$.



Let p_+ be the larger fixed point, $I = [-p_+, p_+]$.

Let $I_0 = \{ x \in I : x < 0 \text{ and } Q_c(x) \in I \}$

and $I_1 = \{ x \in I : x > 0 \text{ and } Q_c(x) \in I \}$.

We have previously defined the Cantor set

$$\Lambda = \{ x \in I : Q_c^n(x) \in I \text{ for all } n \geq 1 \}$$

which has a similar structure as the Cantor Middle Thirds set Γ. Note $\Lambda \subseteq I_0 \cup I_1$.

If $x_0 \in \Lambda$, then $Q_c^n(x_0) \in I_0 \cup I_1$ for all $n \geq 1$.

Let $S(x_0) = (s_0, s_1, s_2, \ldots)$ where

$$s_n = \begin{cases} 0 & \text{if } Q_c^n(x_0) \in I_0 \\ 1 & \text{if } Q_c^n(x_0) \in I_1 . \end{cases}$$

The binary sequence $S(x_0)$ is called the <u>itinerary</u> of x_0.

<u>Definition</u> Suppose $I \subseteq X$ is a subset of phase space such that if $x_0 \in I$, then $F^n(x_0) \in I$ for all $n \geq 1$. Let I_0 and I_1 be disjoint sets such that $I \subseteq I_0 \cup I_1$. Then the <u>itinerary map</u> $S: I \to \{\text{all binary sequences}\}$ is the function defined so that $S(x)$ is a binary sequence $(s_0, s_1, s_2, \dots)$ such that

$$s_n = \begin{cases} 0 & \text{if } F^n(x_0) \in I_0 \\ 1 & \text{if } F^n(x_0) \in I_1 \end{cases}$$

Problem 1. Consider the tent map

$$T(x) = \begin{cases} 3x & x \le 1/2 \\ 3 - 3x & x > 1/2. \end{cases}$$

Let Γ denote the Cantor Middle-Thirds set and let $I_0 = [0, 1/3]$ and $I_1 = [2/3, 1]$. Note that $\Gamma \subset I_0 \cup I_1$. Suppose that $x_0 \in \Gamma$. Then $T^n(x_0) \in \Gamma$ for all $n \ge 1$. Therefore $T^n(x_0) \in I_0 \cup I_1$ for all $n \ge 1$. Let $S(x_0) = (s_0, s_1, s_2, \ldots)$ where

$$s_n = \begin{cases} 0 & \text{if } T^n(x_0) \in I_0 \\ 1 & \text{if } T^n(x_0) \in I_1. \end{cases}$$

a. Find $S(1/3), S(1/4)$, and $S(3/10)$.

b. Bonus: find a period 2 point x_0 such that $S(x_0) = (0, 1, 0, 1, 0, 1, \ldots) = (\overline{01})$.

c. Bonus: find a period 3 point x_0 such that $S(x_0) = (0, 0, 1, 0, 0, 1, \ldots) = (\overline{001})$.

(a) orbit of $1/3$: $\frac{1}{3}, 1, 0, 0, \ldots$

itinerary of $1/3$: $0, 1, 0, 0, \ldots$

orbit of $1/4$: $\frac{1}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \ldots$

itinerary of $1/4$: $0, 1, 1, 1, \ldots$

orbit of $3/10$: $\frac{3}{10}, \frac{9}{10}, \frac{3}{10}, \frac{9}{10}, \ldots$

itinerary of $3/10$: $0, 1, 0, 1, 0, \ldots$

(b) itinerary $\Rightarrow$ $x_n \in I_0$ when n is even,

$x_n \in I_1$ when n is odd

$x_0 \in I_0 \Rightarrow x_1 = 3x_0$

$x_1 \in I_1 \Rightarrow x_2 = 3 - 3x_1 = 3 - 9x_0$

x_0 period 2 $\Rightarrow x_0 = x_2 = 3 - 9x_0 \Rightarrow x_0 = \frac{3}{10}$.