Chapter 9 : Sequence Space and Shift Map

Def The set $\Sigma = \{ (s_0, s_1, s_2, \ldots) : s_n = 0 \text{ or } 1 \text{ for all } n \geq 0 \}$ of all binary sequences is called the sequence space on 2 symbols.

Let $x = (s_0, s_1, \ldots)$, $y = (t_0, t_1, \ldots) \in \Sigma$. The distance between x and y is given by

$$d(x,y) = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i}$$

Example Compute $d(x,y)$ when

1. $x = (0, 0, \ldots) = (\bar{0})$, $y = (1, 1, \ldots) = (\bar{1})$.
2. $x = (111\bar{0})$, $y = (\bar{1})$
3. $x = (1111\bar{0})$, $y = (\bar{1})$

1. $d(x,y) = \displaystyle\sum_{i=0}^{\infty} \frac{1}{2^i} = \frac{1}{1-\frac{1}{2}} = 2$
2. $d(x,y) = \displaystyle\sum_{i=3}^{\infty} \frac{1}{2^i} = \frac{\frac{1}{2^3}}{1-\frac{1}{2}} = \frac{1}{4}$
3. $d(x,y) = \displaystyle\sum_{i=4}^{\infty} \frac{1}{2^i} = \frac{\frac{1}{2^4}}{1-\frac{1}{2}} = \frac{1}{8}$

The more they agree at the start, the closer two sequences are.

Theorem (Proximity Theorem)

Let $x = (s_0, s_1, \ldots)$, $y = (t_0, t_1, \ldots) \in \Sigma$ and $n \in \mathbb{N}$.

① if $s_i = t_i$ for all $i = 0, 1, \ldots, n$, then $d(x,y) \le \frac{1}{2^n}$

② if $d(x,y) < \frac{1}{2^n}$, then $s_i = t_i$ for all $i = 0, \ldots, n$.

Proof

① Suppose $s_i = t_i$ for all $i = 0, \ldots, n$. Then

$$d(x,y) = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i}$$

$$= \sum_{i=n+1}^{\infty} \frac{|s_i - t_i|}{2^i}$$

$$\le \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + \cdots$$

$$= \frac{1/2^{n+1}}{1 - 1/2}$$

$$= \frac{1}{2^n}$$

② To prove a statement of the form $P \Rightarrow Q$ we can instead prove the equivalent contrapositive statement $\neg Q \Rightarrow \neg P$. The contrapositive here is

" if $s_j \ne t_j$ for some $j \in \{0, \ldots, n\}$, then $d(x,y) \ge \frac{1}{2^n}$. "

Suppose $s_j \neq t_j$ for some $j \in \{0, \dots, n\}$. Then

$$d(x,y) = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i}$$

$$\geq \frac{|s_j - t_j|}{2^j}$$

$$= \frac{1}{2^j}$$

$$\geq \frac{1}{2^n}. \quad (\text{since } 2^j \leq 2^n \text{ since } j \leq n.)$$

Def The <u>shift map</u> is the function $\sigma : \Sigma \to \Sigma$ given by $\sigma(s_0, s_1, s_2, \dots) = (s_1, s_2, \dots)$.

We can think of the shift map as defining a dynamical system whose phase space is the sequence space Σ.

<u>Notes</u> about shift map periodic points

1. its fixed points are $(0,0,0,\dots) = (\overline{0})$ and $(1,1,\dots) = (\overline{1})$.
2. its prime period 2 points are $(\overline{01})$ and $(\overline{10})$
3. it has 2^n period n points: $(\overline{x_1 x_2 \cdots x_n})$

   where $x_1, \dots, x_n \in \{0,1\}$.

__Def__ Let $F : \Sigma \to \Sigma$ be a given map. We say F is __continuous__ __at__ $x_0 \in \Sigma$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that: if $y \in \Sigma$ and $d(x, y) < \delta$ then $d(F(x), F(y)) < \varepsilon$.

__Theorem__ The shift map σ is continuous at all $x_0 \in \Sigma$.

__Proof__ Let $x_0 = (s_0, s_1, s_2, \dots) \in \Sigma$ and let $\varepsilon > 0$.

There exists some $n \in \mathbb{N}$ such that $\frac{1}{2^n} < \varepsilon$. Let $\delta = \frac{1}{2^{n+1}}$.

Suppose $y = (t_0, t_1, \dots) \in \Sigma$ such that $d(x_0, y) < \delta$.

Then by the Proximity Theorem $s_0 = t_0, \dots, s_n = t_n, s_{n+1} = t_{n+1}$.

Since $\sigma(x_0) = (s_1, s_2, \dots)$ and $\sigma(y) = (t_1, t_2, \dots)$,

these sequences agree in terms 0 to n (ie $\underbrace{s_1 = t_1}_{\text{0th}}, \dots, \underbrace{s_{n+1} = t_{n+1}}_{n\text{th term}}$)

By the Proximity Theorem, $d(\sigma(x_0), \sigma(y)) \le \frac{1}{2^n} < \varepsilon$.

Problem 1. Let $x = (1,1,1,1,0,0,\ldots) = (1111\overline{0})$ and $y = (1,1,1,\ldots) = (\overline{1})$.

a. Find $d(x,y)$, $d(\sigma(x),\sigma(y))$, and $d(\sigma^4(x),\sigma^4(y))$.

b. What is $d(\sigma^n(x),\sigma^n(y))$ for all $n \geq 5$?

c. Complete the following statement, which is the moral of the story in this problem: x and y are close to each other, but ...

(a) $$d(x,y) = \sum_{n=0}^{\infty} \frac{|s_n - t_n|}{2^n} = \sum_{n=4}^{\infty} \frac{1}{2^n} = \frac{1/2^4}{1-1/2} = \frac{1}{8}$$

$$d(\sigma(x),\sigma(y)) = \sum_{n=3}^{\infty} \frac{1}{2^n} = \frac{1/2^3}{1-1/2} = \frac{1}{4}$$

$$d(\sigma^4(x),\sigma^4(y)) = \sum_{n=0}^{\infty} \frac{1}{2^n} = 2$$

(b) $d(\sigma^n(x),\sigma^n(y)) = 2$ for all $n \geq 5$

(c) x and y are close but $\sigma^n(x)$ and $\sigma^n(y)$ get far apart as n grows

Problem 2. Let $x = (\overline{0})$. Find an example $y \in \Sigma$ such that $d(x,y) < 1/2^5$ and $d(\sigma^7(x),\sigma^7(y)) = 2$.

By the Proximity Theorem if $d(x,y) < \frac{1}{2^5}$ then

$y_0 = x_0 = 0, \ldots, y_5 = x_5 = 0$. Since

$$\sigma^7(y) = (y_7, y_8, \ldots)$$

we need $y_n = 1$ for all $n \geq 7$ in order to have

$$d(\sigma^7(x),\sigma^7(y)) = 2$$

Any y of the form $(\underbrace{0,0,0,0,0,0}_{0 \text{ to } 5}, \underset{\text{free}}{y}, \underbrace{1,1,\ldots}_{7 \text{ onward}})$ works.

Problem 3. Repeat the previous problem except use $x = (\overline{01})$.

$$(\underbrace{0,1,0,1,0,1}_{0 \text{ to } 5}, \underset{\text{free}}{y}, \underbrace{1,0,1,0,\ldots}_{7 \text{ onward}})$$