Chapter 10 Chaos

Definition A dynamical system $F: X \to X$ depends sensitively on initial conditions if there exists $\beta > 0$ such that for every $x \in X$ and every $\varepsilon > 0$ there exists $y \in X$ and $N \in \mathbb{N}$ such that $d(x, y) < \varepsilon$ and $d(F^N(x), F^N(y)) > \beta$.

Theorem The shift map $\sigma: \Sigma \to \Sigma$ depends sensitively on initial conditions.

Proof Define $\beta = 1$. Let $x = (s_0, s_1, s_2, \dots) \in \Sigma$ and $\varepsilon > 0$ be arbitrary. Choose $n \in \mathbb{N}$ so that $\frac{1}{2^n} < \varepsilon$. Define $N = n+1$ and $y = (t_0, t_1, t_2, \dots) \in \Sigma$ as follows.

Let $t_i = s_i$ for $i = 0, \dots, n$ and let $t_i = 1 - s_i$ for all $i \geq n+1$.

Then by the Proximity Theorem, $d(x, y) \leq \frac{1}{2^n} < \varepsilon$. Moreover,

$$\sigma^N(x) = (s_{n+1}, s_{n+2}, \dots) \quad \text{and} \quad \sigma^N(y) = (1 - s_{n+1}, 1 - s_{n+2}, \dots).$$

Therefore $d(\sigma^N(x), \sigma^N(y)) = 2 > \beta$.

Definition Let X be a set and $Y \subseteq X$ a subset. We say Y is **dense** in X if for every $x \in X$ and every $\varepsilon > 0$ there exists $y \in Y$ such that $d(x,y) < \varepsilon$.

Theorem Let $\sigma : \Sigma \to \Sigma$ be the shift map. The periodic points of σ are dense in Σ.

Proof Let $x = (s_0, s_1, s_2, \dots) \in \Sigma$ and $\varepsilon > 0$ be arbitrary. Choose $n \in \mathbb{N}$ s.t. that $\frac{1}{2^n} < \varepsilon$. Define $y \in \Sigma$ as $y = (\overline{s_0 s_1 \dots s_n})$. Then y is a periodic point of σ. Moreover, by the Proximity Theorem, $d(x,y) \leq \frac{1}{2^n} < \varepsilon$.

Definition A dynamical system $F : X \to X$ is **transitive** if for any $x, y \in X$ and any $\varepsilon > 0$ there exists $z \in X$ and $n \in \mathbb{N}$ such that $d(x,z) < \varepsilon$ and $d(F^n(z), y) < \varepsilon$.

Definition A dynamical system $F: X \to X$ has a **dense orbit** if there exists $z \in X$ such that $\{F^n(z) : n \geq 0\}$ is dense in X.

Lemma If a dynamical has a dense orbit then it is transitive.

Proof Exercise.

Theorem The shift map $\sigma : \Sigma \to \Sigma$ has a dense orbit.

Proof Let $z \in \Sigma$ be defined as

$$z = (\underbrace{0\ 1}_{1 \text{ blocks}}\ \underbrace{00\ 01\ 10\ 11}_{2 \text{ blocks}}\ \underbrace{000\ 001\ \cdots}_{3 \text{ blocks}}).$$

In words, z is a sequence that consists of all possible length 1 sequences, then all possible length 2 sequences, then all possible length 3 sequences, and so on. We claim the orbit of z, $\{\sigma^n(z) : n \geq 0\}$ is dense in Σ.

Let $x = (s_0, s_1, s_2, \ldots) \in \Sigma$ and $\varepsilon > 0$ be arbitrary. We claim there exists $N \in \mathbb{N}$ such that $d(x, \sigma^N(z)) < \varepsilon$. Choose $n \in \mathbb{N}$ so that $\frac{1}{2^n} < \varepsilon$. Within z there is a block of length $n+1$ which consists of $s_0, s_1, \ldots, s_n$ (the first $n+1$ elements of x). Let N be the entry of z where this block starts. Then

$$d(x, \sigma^N(z)) \leq \frac{1}{2^n} < \varepsilon$$

by the Proximity Theorem.

Definition A dynamical system $F: X \to X$ is called chaotic if

① it depends sensitively on initial conditions

"unpredictible"

② its periodic points are dense

"but with some regularity"

③ it is transitive

"cannot be decomposed into independent subsystems"

Theorem The shift map $\sigma: \Sigma \to \Sigma$ is chaotic.