Chapters 9,10 Chaos and Conjugacy

Def A function $h: X \to Y$ is continuous at $x_0 \in X$ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that: if $y \in X$ and $d(x_0, y) < \delta$ then $d(h(x), h(y)) < \varepsilon$. The function is called continuous if it is continuous at all $x_0 \in X$. The function is called a homeomorphism if it is continuous, invertible, and its inverse is continuous.

Def Let $F: X \to X$ and $G: Y \to Y$ be two dynamical systems with phase spaces X and Y respectively.
We say F and G are topologically conjugate if there exists a homeomorphism $h: X \to Y$ such that
$G = h \circ F \circ h^{-1}$ (or equivalently, $G \circ h = h \circ F$)

The function h is called a conjugacy.

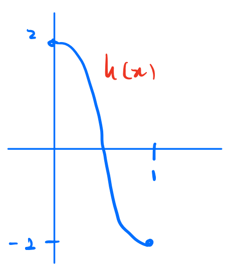
$$\begin{tikzcd} X \arrow[r, "F"] \arrow[d, "h"'] & X \arrow[d, "h"] \\ Y \arrow[r, "G"'] & Y \end{tikzcd}$$

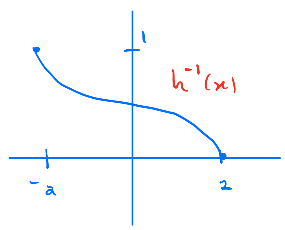
Example The quadratic map $Q : [-2,2] \to [-2,2]$, $Q(x) = x^2 - 2$

and the tent map $T : [0,1] \to [0,1]$, $T(x) = \begin{cases} 2x & 0 \le x \le \frac{1}{2} \\ 2-2x & \frac{1}{2} \le x \le 1 \end{cases}$

are topologically conjugate.

Let $h : [0,1] \to [-2,2]$ be $h(x) = 2\cos(\pi x)$.





continuous, invertible,

and inverse $h^{-1}(x) = \frac{1}{\pi}\cos^{-1}\left(\frac{x}{2}\right)$

is continuous

Then $h \circ T(x) = \begin{cases} 2\cos(2\pi x) & 0 \le x \le \frac{1}{2} \\ \underbrace{2\cos(2\pi - 2\pi x)} & \frac{1}{2} \le x \le 1 \end{cases} = 2\cos(2\pi x)$ for all $x \in [0,1]$

$= 2\cos(-2\pi x)$ since $\cos(\theta + 2\pi) = \cos\theta$

$= 2\cos(2\pi x)$ since $\cos(-\theta) = \cos\theta$

and $Q \circ h(x) = 4\cos^2(\pi x) - 2$

$= 2\left(2\cos^2(\pi x) - 1\right)$

$= 2\cos(2\pi x)$ since $2\cos^2\theta - 1 = \cos(2\theta)$

$= h \circ T(x)$.

Theorem Let $c < -2$. The quadratic map $Q_c : \Lambda \to \Lambda$ and the shift map $\sigma : \Sigma \to \Sigma$ are topologically conjugate. The itinerary map $S : \Lambda \to \Sigma$ is a conjugacy.

Proof See section 9.4. We won't have time for it.

What is preserved by conjugacy?

- Orbit structure : • if $(x_0, x_1, x_2, \dots)$ is the orbit of x_0 under F, then $(h(x_0), h(x_1), h(x_2), \dots)$ is the orbit of $h(x_0)$ under G.
  - if $\lim_{n\to\infty} F^n(x_0) = \alpha$, then $\lim_{n\to\infty} G^n(h(x_0)) = h(\alpha)$.
- Periodic points, fixed points

  If x_0 is a fixed point of F then

  $$G(h(x_0)) = h(F(x_0)) = h(x_0)$$

  implies $h(x_0)$ is a fixed point of G

- Dense sets

Lemma Let $F: X \to X$ and $G: Y \to Y$ be topologically conjugate dynamical systems with conjugacy $h: X \to Y$. If $A \subseteq X$ is dense in X, then $h(A) = \{h(a) : a \in A\}$ is dense in Y.

Proof Let $y \in Y$ and $\varepsilon > 0$ be arbitrary. We must show there exists $z \in h(A)$ such that $d(z, y) < \varepsilon$. Since h is continuous there exists $\delta > 0$ such that for any $x \in X$, $d(x, h^{-1}(y)) < \delta$ implies $d(h(x), y) < \varepsilon$. Since A is dense there exists $a \in A$ such that $d(a, h^{-1}(y)) < \delta$. Let $z = h(a)$. Then $z \in h(A)$ and $d(z, y) < \varepsilon$.

Lemma Let $F: X \to X$ and $G: Y \to Y$ be topologically conjugate dynamical systems with conjugacy $h: X \to Y$. If $x_0 \in X$ is a periodic point of F, then $h(x_0)$ is a periodic point of G.

Proof Homework

Theorem Let $F: X \to X$ and $G: Y \to Y$ be topologically conjugate dynamical systems with conjugacy $h: X \to Y$.

1. if F depends sensitively on initial conditions then G depends sensitively on initial conditions.

   (under an extra technical assumption about the phase space being what is called a compact set)

2. if the periodic points of F are dense in X then the periodic points of G are dense in Y

3. if F has a dense orbit, then G has a dense orbit

Corollary If F and G are topologically conjugate, then F being chaotic implies G is chaotic.