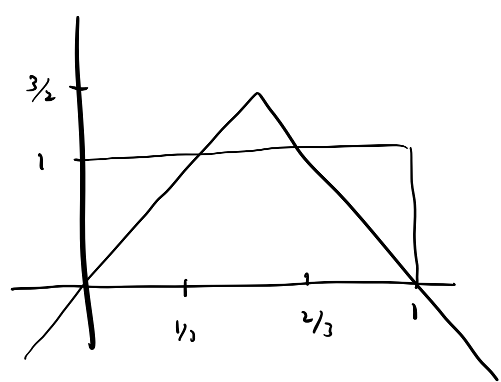
Dynamics on the Cantor-Middle Thirds Set

Let's consider the map

$$T(x) = \begin{cases} 3x & \text{if } x \le \frac{1}{2} \\ 3-3x & \text{if } x > \frac{1}{2} \end{cases}$$



In homework, I asked you to investigate orbits in $[0,1]-K$ and you saw that if $x_0 \in [0,1]-K$, $T^n(x_0) \to -\infty$ as $n \to \infty$.

Question What does T do to elements of K?

Given $x_0 \in K$, what can you say about the ternary expansion of $T(x_0)$?

Discussion Suppose $x_0 \in K$ and $x_0 \leq \frac{1}{3}$. Then its ternary expansion is given by

$$x_0 = 0.0\, s_2 s_3 s_4 \ldots$$

for some $s_2, s_3, s_4, \ldots$. Then

$$
\begin{aligned}
T(x_0) &= 3x_0 \qquad \text{since } x_0 \leq \tfrac{1}{2} \\
&= 3\left(\frac{s_2}{3^2} + \frac{s_3}{3^3} + \frac{s_4}{3^4} + \cdots \right) \\
&= \frac{s_2}{3} + \frac{s_3}{3^2} + \frac{s_4}{3^5} + \cdots \\
&= 0.s_2 s_3 s_4 \ldots
\end{aligned}
$$

So T shifts the ternary expansion of x_0

Suppose $x_0 \in K$ and $x_0 \geq \frac{2}{3}$. Then

$$x_0 = 0.2 s_2 s_3 s_4 \ldots$$

and $T(x_0) = 3 - 3x_0$ since $x > \frac{1}{2}$

$$= 3 - 3\left(\frac{2}{3} + \frac{s_2}{3^2} + \frac{s_3}{3^3} + \frac{s_4}{3^4} + \cdots\right)$$

$$= 3 - \left(2 + \frac{s_2}{3} + \frac{s_3}{3^2} + \frac{s_4}{3^3} + \cdots\right)$$

$$= 1 - \left(\frac{s_2}{3} + \frac{s_3}{3^2} + \frac{s_4}{3^3} + \cdots\right)$$

$$= \left(\frac{2}{3} + \frac{2}{3^2} + \frac{2}{3^3} + \cdots\right) - \left(\frac{s_2}{3} + \frac{s_3}{3^2} + \frac{s_4}{3^3} + \cdots\right)$$

$$= \frac{2-s_2}{3} + \frac{2-s_3}{3^2} + \frac{2-s_4}{3^3} + \cdots$$

$$= 0.(2-s_2)(2-s_3)(2-s_4)\ldots$$

So T flips 0's and 2's and shifts the ternary expansion of x_0.

Question ① Use the ternary expansion $3/4$ to explain why it's a fixed point of T.

② What can you say about $\frac{1}{4} = 0.\overline{02}$? Is it a fixed point of T?

③ There 2 period-2 points of T.

Can you show $0.\overline{0220}$ is one of them?

What is the other?

④ Can you find a 3-cycle of T?

This is not that easy!