**Problem 1.** The following questions are about ternary expansions and the map

$$T(x) = \begin{cases} 3x & x \le 1/2 \\ 3 - 3x & x > 1/2. \end{cases}$$

a. Find real numbers whose ternary expansions are:

1. $0.001\overline{0}$
2. $0.00\overline{2}$
3. $0.0\overline{101}$
4. $0.\overline{20}$

b. Are any of the above elements of the Cantor middle-thirds set $\Gamma$? Are any of them endpoints of $\Gamma$?

c. Find a ternary expansion of each prime period 2 point of $T$.

d. Suppose $x \in \Gamma$ with ternary expansion $0.a_1a_2a_3\ldots$. What is the ternary expansion of $T(x)$? Note that you'll have to consider two cases. In one of the two cases it will be helpful to note that $\sum_{n=1}^{\infty} 2/3^n = 1$.

(a)

(1) $\frac{1}{3^3} = \frac{1}{27} \in \Gamma$ and endpoint

(2) $\sum_{n=3}^{\infty} \frac{2}{3^n} = \frac{2/27}{1-1/3} = \frac{2/27}{2/3} = \frac{1}{9} \in \Gamma$ and endpoint

(3) $\frac{1}{3^2} + \frac{1}{3^4} + \frac{1}{3^5} + \frac{1}{3^7} + \cdots$

$= \left(\frac{1}{3^2} + \frac{1}{3^5} + \frac{1}{3^8} + \cdots\right) + \left(\frac{1}{3^4} + \frac{1}{3^7} + \frac{1}{3^{10}} + \cdots\right)$

$= \frac{1/9}{1-1/27} + \frac{1/81}{1-1/27}$

$= \frac{27}{26}\left(\frac{9}{81} + \frac{1}{81}\right)$

$= \frac{1}{26}\left(\frac{10}{3}\right) = \frac{5}{39} \notin \Gamma$

(4) $\frac{2}{3} + \frac{2}{3^3} + \frac{2}{3^5} = \frac{2/3}{1-1/9} = \frac{2}{3} \cdot \frac{9}{8} = \frac{3}{4} \in \Gamma$ but not endpoint

(c) A point $x_0$ with itinerary $(\overline{01})$ has prime period 2.

$x_1 = T(x_0) = 3x_0$

$x_0 = x_2 = T(x_1) = 3 - 3x_1 = 3 - 9x_0$

$\Rightarrow \quad x_0 = 3/10$

Ternary expansion of $3/10$ :

$$\frac{3}{10} = \frac{x_1}{3} + \frac{x_2}{3^2} + \frac{x_3}{3^3} + \dots$$

$$\frac{9}{10} = x_1 + \frac{x_2}{3} + \frac{x_3}{3^2} + \dots \quad \Rightarrow \quad x_1 = 0$$

$$2 + \frac{7}{10} = \frac{27}{10} = x_2 + \frac{x_3}{3} + \frac{x_4}{3^2} + \dots \quad \Rightarrow \quad x_2 = 2$$

$$\frac{7}{10} = \frac{x_3}{3} + \frac{x_4}{3^2} + \dots$$

$$2 + \frac{1}{10} = \frac{21}{10} = x_3 + \frac{x_4}{3} + \frac{x_5}{3^2} + \dots \quad \Rightarrow \quad x_3 = 2$$

$$\frac{1}{10} = \frac{x_4}{3} + \frac{x_5}{3^2} + \dots$$

$$\frac{3}{10} = x_4 + \frac{x_5}{3} + \frac{x_6}{3^2} + \dots \quad \Rightarrow \quad x_4 = 0$$

$$\frac{3}{10} = 0.\overline{0220}$$

Ternary expansion of $9/10$ : $\frac{9}{10} = 0.\overline{2200}$

(d) if $a_1 = 0$, then $x \le \frac{1}{2}$ and

$$T(x) = 3x = 3\left(\frac{a_1}{3} + \frac{a_2}{3^2} + \frac{a_3}{3^3} + \dots\right)$$

$$= \frac{a_2}{3} + \frac{a_3}{3^2} + \frac{a_4}{3^3} + \dots$$

which means $T(x)$ has ternary expansion $0.a_2a_3a_4\dots$

if $a_1 = 2$, then $x > \frac{1}{2}$ and so

$$T(x) = 3 - 3x$$

$$= 3 - 3\left(\frac{a_1}{3} + \frac{a_2}{3^2} + \frac{a_3}{3^3} + \cdots\right)$$

$$= 1 - \left(\frac{a_2}{3} + \frac{a_3}{3^2} + \cdots\right)$$

$$= \left(\frac{2}{3} + \frac{2}{3^2} + \frac{2}{3^3} + \cdots\right) - \left(\frac{a_2}{3} + \frac{a_3}{3^2} + \frac{a_4}{3^3} + \cdots\right)$$

$$= \frac{2-a_2}{3} + \frac{2-a_3}{3^2} + \frac{2-a_4}{3^3} + \cdots$$

which means $T(x)$ has ternary expansion

$$0.(2-a_2)(2-a_3)(2-a_4)\ldots$$

**Problem 2.** The following questions are about the sequence space on two symbols

$$\Sigma = \{(x_0, x_1, x_2, \ldots) : x_n \in \{0,1\} \text{for all } n \geq 0\}$$

with distance function $d$ given by

$$d(x,y) = \sum_{n=0}^{\infty} \frac{|x_n - y_n|}{2^n}$$

and the shift map $\sigma : \Sigma \to \Sigma$ given by $\sigma(x_0, x_1, x_2, \ldots) = (x_1, x_2, \ldots)$.

a. Find the distance $d(x,y)$ when

1. $x = (\overline{100}), y = (\overline{010})$
2. $x = (\overline{1011}), y = (\overline{01})$

b. Find the two sequences which are midway between $x = (\overline{0})$ and $y = (\overline{1})$. In other words, find $z_1$ and $z_2$ such that $d(x, z_i) = d(y, z_i) = d(x,y)/2$ for $i = 1, 2$.

c. Which element of $\Sigma$ is furthest from the sequence $(s_0 s_1 s_2 \ldots)$?

d. Let $x = (\overline{0})$. Find $y \in \Sigma$ which is an eventually fixed point of $\sigma$ such that $d(x,y) \leq 1/2^3$.

e. Let $x = (\overline{0})$. Find $y \in \Sigma$ and $N \in \mathbb{N}$ such that $d(x,y) < 1/2^3$ and $d(\sigma^N(x), \sigma^N(y)) = 2$.

(a) (1) $d(x,y) = \frac{1}{2^0} + \frac{1}{2^1} + \frac{0}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{0}{2^5} + \cdots$

$= \left(\frac{1}{2^0} + \frac{1}{2^3} + \frac{1}{2^6} + \cdots\right) + \left(\frac{1}{2^1} + \frac{1}{2^5} + \frac{1}{2^8} + \cdots\right)$

$= \frac{1}{1 - 1/2^3} + \frac{1/2^1}{1 - 1/2^3}$

$= \frac{8}{7} + \frac{1}{2} \cdot \frac{8}{7}$

$= \frac{12}{7}$

(2) $d(x,y) = \frac{1}{2^0} + \frac{1}{2^1} + \frac{1}{2^2} + \frac{0}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6} + \frac{0}{2^7} + \cdots$

$= \frac{1}{1 - 1/16} + \frac{1/2}{1 - 1/16} + \frac{1/4}{1 - 1/16}$

$= \frac{16}{15}\left(1 + \frac{1}{2} + \frac{1}{4}\right)$

$= \frac{16}{15}\left(\frac{7}{4}\right)$

$= \frac{28}{15}$

(b) $z_1 = (1, 0, 0, \dots)$ $\quad$ $z_2 = (0, 1, 1, \dots)$

(c) $(1-s_0, 1-s_1, 1-s_2, \dots)$

(d) $y = (0, 0, 0, 1, \overline{0})$

$d(x,y) = \frac{1}{2^3}$

(e) $y = (0, 0, 0, 0, 0, \overline{1})$, $N = 5$

$$d(x,y) = \sum_{n=5}^{\infty} \frac{1}{2^n} = \frac{1/2^5}{1-1/2} = \frac{1}{2^4} < \frac{1}{2^3}$$

$$d(\sigma^N(x), \sigma^N(y)) = \sum_{n=0}^{\infty} \frac{1}{2^n} = 2.$$

**Problem 3.** State whether each of the following is true or false.

a. The set $Y = \{(s_0, s_1, s_2, \ldots) \in \Sigma : s_4 = 0\}$ is dense in $\Sigma$.

b. Consider $x = 9/10 \in \Gamma$. The itinerary of $x$ is just its ternary expansion with each digit divided by 2.

c. Suppose $F : X \to X$ and $G : Y \to Y$ are conjugate dynamical systems with conjugacy $h : X \to Y$. If $x_0 \in X$ is an eventually fixed point of $F$ then $h(x_0)$ is an eventually fixed point of $G$.

d. The shift map has 30 prime period 5 points.

(a) False. In order for $Y$ to be dense we need that for every $x \in \Sigma$ and every $\varepsilon > 0$, there exists $y \in Y$ such that $d(x,y) < \varepsilon$.

Consider $x = (1,1,1,\ldots) = (\overline{1})$ and $\varepsilon = \frac{1}{4}$

Then since every $y \in Y$ is of the form

$$(s_0, s_1, 0, s_3, \ldots)$$

we have for any $y \in Y$,

$$d(x,y) = \sum_{n=0}^{\infty} \frac{1-s_n}{2^n}$$

$$= \frac{1-s_0}{2^0} + \frac{1-s_1}{2^1} + \frac{1}{2^2} + \frac{1-s_3}{2^3} + \cdots$$

$$\geq \frac{1}{4}$$

$$= \varepsilon.$$

Thus no element of $Y$ can be closer than $1/4$ from $x$.

(b) False. See Problem 1.

(c) True. For $x_0$ to be an eventually point means it is not a fixed point of $F$, but $y = F^N(x_0)$ is a fixed point of $F$ for some $N \in \mathbb{N}$. Then $h(x_0)$ is not a fixed point of $G$ (otherwise this would mean $x_0$ is a fixed point of $F$). Moreover we claim $G^N(h(x_0))$ is a fixed point of $G$. Indeed, note that $G^N(h(x_0)) = h(F^N(x_0)) = h(y)$. Since $y$ is a fixed point of $F$, $h(y)$ is a fixed point of $G$.

(d) True. The period 5 points are all sequences of the form $(\overline{s_0 s_1 s_2 s_3 s_4})$. There are $2^5 = 32$ of these. Removing the 2 fixed points gives 30 prime period 5 points.