Chapter 3 More Basics of Orbits

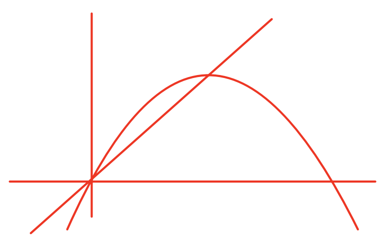
Example Let $F(x) = 2x(1-x)$. We saw last time that the orbit of $x_0 = 0$ was constant. That is, $F^n(0) = 0$ and so $x_0 = x_1 = x_2 = \dots = 0$. Same with $x_0 = \frac{1}{2}$. Are there any other orbits like this?

Def A fixed point of F is an element $x_0 \in X$ of phase space that solves the equation $F(x) = x$.

(Note the orbit of a fixed point is a constant sequence $x_0 = x_1 = x_2 = \dots$)

Example The fixed points of $F(x) = 2x(1-x)$ are solutions of $2x(1-x) = x$, which is equivalent to the equation $x - 2x^2 = 0$, which only has 2 solutions $x = 0$ and $x = \frac{1}{2}$. So these are the only fixed points.

Note solutions of $F(x) = x$ are the intersection points of the graphs of $y = F(x)$ and $y = x$.



Def A point $x_0 \in X$ is periodic if $F^n(x_0) = x_0$ for some $n \geq 1$. The smallest such n is called the prime period of x_0 and the orbit

$$x_0, F(x_0), F^2(x_0), \dots, F^{n-1}(x_0), x_0, F(x_0), \dots$$

is called an n-cycle or a periodic orbit.

Example When $F(x) = x^2 - 1$, $x_0 = 0$ is a periodic point of prime period 2. The orbit of x_0 is

$$0, -1, 0, -1, 0, \dots$$

and is called a 2-cycle.

Def A point $x_0 \in X$ is called eventually periodic (or eventually fixed) if it is not a periodic point (or fixed point) but some element of its orbit is.

Example $x_0 = 1$ is an eventually periodic point for $F(x) = x^2 - 1$.

Example (the doubling map) Let $X = [0,1)$ and let $D : X \to X$ be the map

$$D(x) = \begin{cases} 2x & 0 \le x < \frac{1}{2} \\ 2x-1 & \frac{1}{2} \le x < 1 \end{cases}$$

Sometimes, D is denoted as $D(x) = 2x \pmod 1$.

Notice that $D(x)$ takes the fractional part of $2x$:

$D(0.3) = 0.6$, $D(0.6) = 1.2 - 1 = 0.2$

Compute the orbits of $\frac{1}{5}$ and $\frac{1}{9}$.

① $\frac{1}{5}, \frac{2}{5}, \frac{4}{5}, \frac{8}{5}, \frac{16}{5}, \frac{32}{5}, \frac{64}{5}, \frac{128}{5}$

$\frac{8}{5} = 1+\frac{3}{5}$, $\frac{16}{5} = 3+\frac{1}{5}$, $\frac{32}{5} = 6+\frac{2}{5}$, $\frac{64}{5} = 12+\frac{4}{5}$, $\frac{128}{5} = 25+\frac{3}{5}$

$\frac{1}{5}, \frac{2}{5}, \frac{4}{5}, \frac{3}{5}, \frac{1}{5}, \frac{2}{5}, \frac{4}{5}, \frac{3}{5}, \ldots$ (a 4-cycle)

② $\frac{1}{9}, \frac{2}{9}, \frac{4}{9}, \frac{8}{9}, \frac{16}{9}, \frac{32}{9}, \frac{64}{9}, \frac{128}{9}, \frac{256}{9}$

$\frac{16}{9} = 1+\underline{\frac{7}{9}}$, $\frac{32}{9} = 3+\underline{\frac{5}{9}}$, $\frac{64}{9} = 7+\underline{\frac{1}{9}}$, $\frac{128}{9} = 14+\underline{\frac{2}{9}}$, $\frac{256}{9} = 28+\underline{\frac{4}{9}}$

$\frac{1}{9}, \frac{2}{9}, \frac{4}{9}, \frac{8}{9}, \frac{7}{9}, \frac{5}{9}, \frac{1}{9}, \frac{2}{9}, \ldots$ (a 6-cycle)

Theorem If $x_0 = \frac{k}{2^m} \in [0, 1)$ for some $k, m \in \mathbb{N}$, then x_0 is an eventually fixed point.

Proof Since $2^m x_0 = k$, it follows that $D^m(x_0) = 0$. Therefore, since $D(0) = 0$, x_0 is eventually fixed after m iterates of D.

Theorem If $x_0 = \frac{k}{2^m - 1} \in [0, 1)$ for some $k, m \in \mathbb{N}$, then x_0 is periodic of period m.

Proof Exercise.