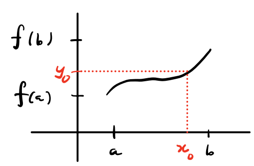
Chapter 5 Fixed and Periodic Points

Is there a condition on F that guarantees a fixed point to exist?

Theorem (Intermediate Value Theorem) Let $a, b \in \mathbb{R}$ with $a < b$ and let $f : [a,b] \to \mathbb{R}$ be a continuous function. Then for every y_0 between $f(a)$ and $f(b)$, there exists $x_0 \in [a,b]$ such that $f(x_0) = y_0$.



Corollary (Fixed Point Theorem) Let $F : [a,b] \to [a,b]$ be continuous. Then there exists $x_0 \in [a,b]$ such that

$$F(x_0) = x_0.$$

Proof Let $g(x) = F(x) - x$. Note that $F(a) \in [a,b]$ and therefore $F(a) \geq a$. Similarly, $F(b) \leq b$. Thus,

$$g(a) = F(a) - a \geq 0$$

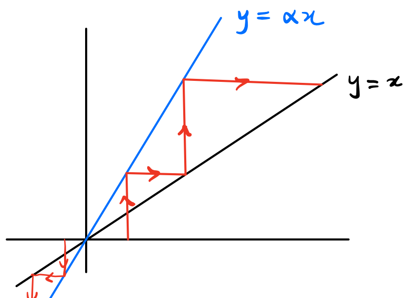
and $$g(b) = F(b) - b \leq 0.$$

This means 0 is between $g(a)$ and $g(b)$. By the Intermediate Value Theorem, there exists $x_0 \in [a,b]$ such that $g(x_0) = 0$. Therefore $F(x_0) = x_0$.

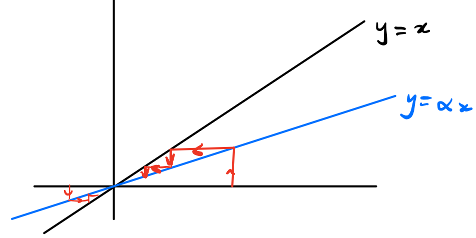
Note not every map we've seen is continuous (doubling map is not). Also the theorem only proves existence of fixed point, and it does not give a way to find them or explain how many there are.

Attraction and Repulsion

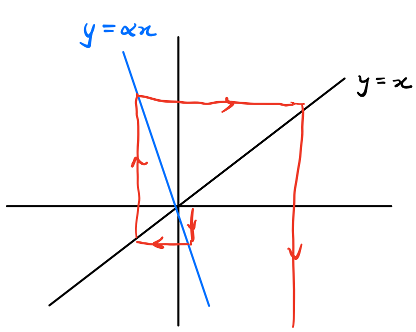
Example Consider $F(x) = \alpha x$ when $\alpha \in \mathbb{R}$ is a given constant.



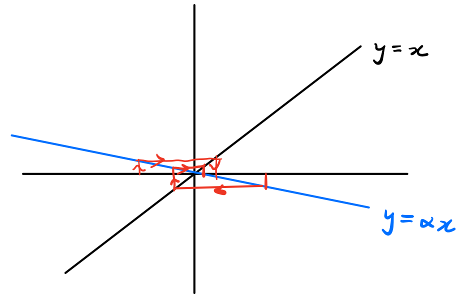
$\alpha > 1$



$0 < \alpha < 1$



$\alpha < -1$



$-1 < \alpha < 0$

The pictures above show that when our orbits tend to attract toward the fixed point $x_0 = 0$ when $|\alpha| < 1$ and repel away when $|\alpha| > 1$.

Definition Suppose x_0 is a fixed point of a map F. Then x_0 is an attracting fixed point if $|F'(x_0)| < 1$. We call x_0 a repelling fixed point if $|F'(x_0)| > 1$. We call x_0 a neutral fixed point if $|F'(x_0)| = 1$.

Example Let $F(x) = 2x(1-x)$. Then $x_0 = \frac{1}{2}$ and $x_0 = 0$ are fixed points. Also $F'(x) = 2 - 4x$. So

$|F'(\frac{1}{2})| = 0 < 1$ implies $\frac{1}{2}$ is attracting and

$|F'(0)| = 2 > 1$ implies 0 is repelling.

Example $F_1(x) = x - x^2$, $F_2(x) = x - x^3$, $F_3(x) = x + x^3$

$x_0 = 0$ is a fixed point for each of these maps. Is it attracting, repelling, or neutral? Do orbits with initial seeds close to 0 converge to 0?