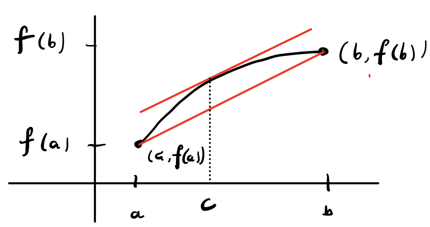
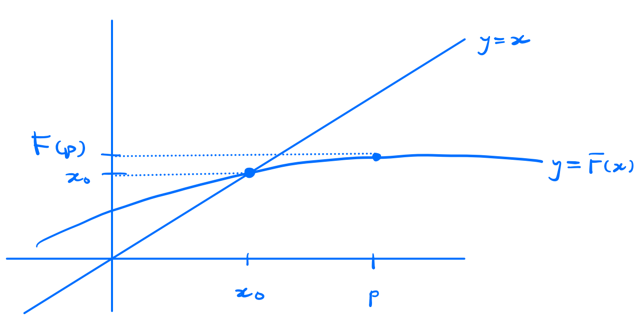
Chapter 5 Fixed and Periodic Points

Now our goal is to understand why orbits that start near an attracting fixed point converge to it. We have some intuition from graphical analysis, but we'd like a more rigorous argument.

Theorem (Mean Value Theorem) Let f be a function that is differentiable at all $x \in [a,b]$. Then there exists $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b-a}.$$





Theorem (Attracting Fixed Point Theorem) Suppose x_0 is an attracting fixed point of F. Then there exists an interval I which contains x_0 in its interior (ie x_0 is not an endpoint of I) such that for every $p \in I$, $F^n(p) \in I$ for all $n \geq 1$ and $\lim_{n \to \infty} F^n(p) = x_0$.

We'll also suppose F' is continuous at x_0.

Proof Since $|F'(x_0)| < 1$, there exists $\lambda > 0$ such that $|F'(x_0)| < \lambda < 1$. There also exists an interval I containing x_0 such that $|F'(x)| < \lambda$ for all $x \in I$. (Otherwise F' would have a jump discontinuity at x_0). Let $p \in I$. By the Mean Value Theorem

$$\left| \frac{F(p) - \overset{x_0}{\cancel{F(x_0)}}}{p - x_0} \right| = |F'(c_1)| < \lambda$$

for some c_1 between p and x_0, which implies

$$|F(p) - x_0| < \lambda |p - x_0| < |p - x_0|.$$

This means $F(p) \in I$ since $F(p)$ is closer to x_0 than p is to x_0.

By the Mean Value Theorem

$$\left| \frac{F^2(p) - \underset{}{F^2(x_0)}^{= x_0}}{F(p) - F(x_0)} \right| = |F'(c_2)| < \lambda$$

for some c_2 between $F(p)$ and $F(x_0)$.

This implies

$$\begin{aligned} |F^2(p) - x_0| &< \lambda\, |F(p) - F(x_0)| \\ &< \lambda^2 |p - x_0| \end{aligned}$$

Again we see $F^2(p) \in I$. Repeating this argument, we get $F^n(p) \in I$ and

$$0 \le |F^n(p) - x_0| < \lambda^n |p - x_0|$$

for all $n \ge 1$. By the Squeeze Theorem

$$\lim_{n \to \infty} |F^n(p) - x_0| = 0$$

since $\lim\limits_{n \to \infty} \lambda^n = 0$. This implies $\lim\limits_{n \to \infty} F^n(p) = x_0$.