Chapter 6 Bifurcations

For each $c \in \mathbb{R}$, let $Q_c : \mathbb{R} \to \mathbb{R}$ be given by $Q_c(x) = x^2 + c$. We call Q_c the quadratic map.

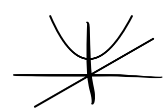
Solutions to $Q_c(x) = x$ (ie. $x^2 - x + c = 0$)

are given by

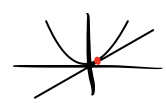
$$p_{\pm} = \frac{1 \pm \sqrt{1-4c}}{2}$$

using the quadratic formula. This means

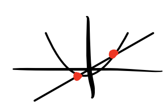
- when $1-4c < 0$ (ie. $c > \frac{1}{4}$),
  Q_c has no fixed points



- when $1-4c = 0$ (ie. $c = \frac{1}{4}$),
  Q_c has $\underline{1}$ fixed point, $p = \frac{1}{2}$



- when $1-4c > 0$, (ie. $c < \frac{1}{4}$),
  Q_c has 2 fixed points, $p_{\pm}$



This sudden transition from 0 to 1 to 2 fixed points is called a bifurcation. For reasons we can discuss later, this is called a saddle node bifurcation (spoiler: when we go from 1 neutral fixed point to 2 fixed points with one attracting and one repelling, in analogy with saddle surfaces like $f(x,y) = x^2 - y^2$)

When there is 1 fixed point ($p = \frac{1}{2}$) this fixed point is neutral since $Q_c'(x) = 2x$ and so $|Q_c'(\frac{1}{2})| = 1$.

When there are 2 fixed points,

$$|Q_c'(p_+)| = 2p_+ = 1 + \sqrt{1-4c} > 1$$

which tells us p_+ is always repelling. Also,

$$|Q_c'(p_-)| < 1 \quad \text{if and only if}$$

$$-1 < 1 - \sqrt{1-4c} < 1$$

$$\iff \quad -2 < -\sqrt{1-4c} < 0$$

$$\iff \quad 0 < \sqrt{1-4c} < 2$$

$$\iff \quad 0 < 1-4c < 4$$

$$\iff \quad -1 < -4c < 3$$

$$\iff \quad -3 < 4c < 1$$

$$\iff \quad -\frac{3}{4} < c < \frac{1}{4}.$$

Therefore, p_- is attracting when $-\frac{3}{4} < c < \frac{1}{4}$.

neutral when $c = -\frac{3}{4}$,

and repelling when $c < -\frac{3}{4}$.

Period 2 points We can solve the equation

$$Q_c^2(x) = x$$

to find prime period 2 points. Note

$$Q_c^2(x) = (x^2+c)^2 + c$$

$$= x^4 + 2cx^2 + c^2 + c$$

and

$$0 = Q_c^2(x) - x$$

$$= x^4 + 2cx^2 - x + c^2 + c$$

$$= (x - p_+)(x + p_-)\, g(x)$$

$$= (x^2 - x + c)\, g(x)$$

where $g(x)$ is a degree 2 polynomial. Note that by polynomial long division,

$$g(x) = \frac{x^4 + 2cx^2 - x + c^2 + c}{x^2 - x + c}$$

$$= x^2 + x + c + 1$$

Therefore, we can solve $g(x) = 0$ to get

$$q_\pm = \frac{-1 \pm \sqrt{1-4(c+1)}}{2}$$

$$= \frac{-1 \pm \sqrt{-4c-3}}{2}$$

using the quadratic formula. So

- when $-4c-3 \leq 0$ (ie $c \geq \frac{-3}{4}$), Q_c has no 2-cycles
- when $c < \frac{-3}{4}$ has a 2-cycle on which $q_\pm$ lie

Moreover,

$$
\begin{aligned}
|(Q^2)'(q_\pm)| &= |Q'(q_+)||Q'(q_-)| \\
&= 4|q_+ \cdot q_-| \\
&= |(-1+\sqrt{-4c-3})(-1-\sqrt{-4c-3})| \\
&= |1-(-4c-3)| \\
&= |4c+4|
\end{aligned}
$$

Therefore,

$$|(Q^2)'(q_\pm)| < 1$$

$$\iff \quad -1 < 4c+4 < 1$$

$$\iff \quad -5 < 4c < -3$$

$$\iff \quad -\frac{5}{4} < c < -\frac{3}{4}$$

So

- the 2-cycle is attracting when $-\frac{5}{4} < c < -\frac{3}{4}$
- the 2-cycle is repelling when $c < -\frac{5}{4}$.