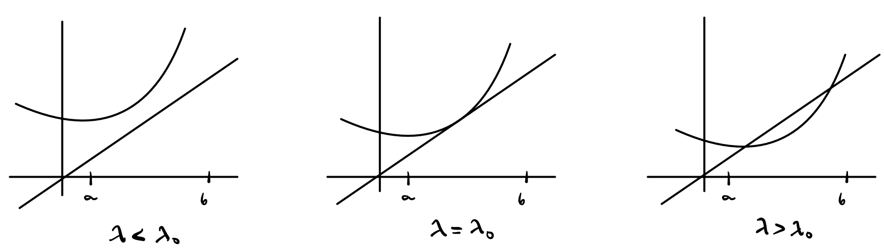
Chapter 6 More on bifurcations

Def A one-parameter family of maps $F_\lambda : X \to X$ undergoes a saddle-node bifurcation (also called a tangent bifurcation) at the parameter value λ_0 (called the bifurcation value) if there is an open interval $I \subseteq X$ and $\varepsilon > 0$ such that

(1) when $\lambda \in (\lambda_0 - \varepsilon, \lambda_0)$, F_λ has no fixed points in I

(2) when $\lambda = \lambda_0$, F_λ has one fixed point in I and this fixed point is neutral

(3) when $\lambda \in (\lambda_0, \lambda_0 + \varepsilon)$ has two fixed points in I, one attracting and one repelling.

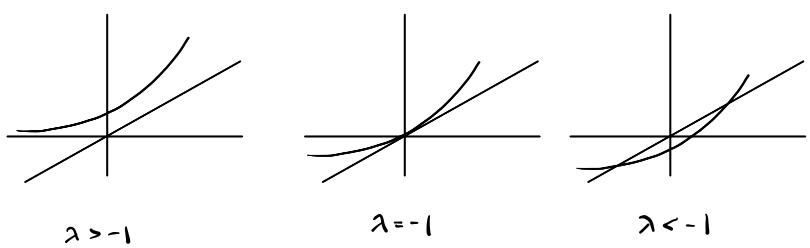


Examples

① The quadratic map $Q_c(x) = x^2 + c$ undergoes a saddle node bifurcation at $c = \frac{1}{4}$.

No fixed points when $c > \frac{1}{4}$, one fixed point when $c = \frac{1}{4}$, and an attracting and a repelling fixed point when $-\frac{3}{4} < c < \frac{1}{4}$ (so $\varepsilon = 1$, $I = \mathbb{R}$)

② The map $E_\lambda(x) = e^x + \lambda$ undergoes a saddle-node bifurcation when $\lambda = -1$.

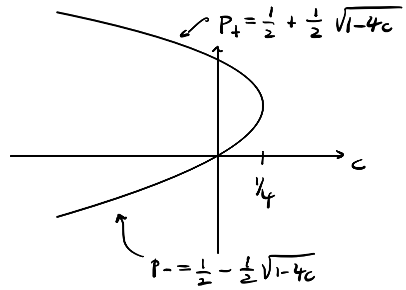


Note that finding fixed points when $\lambda < -1$ cannot be done algebraically

Remark In these examples the bifurcation occurs as we decrease λ instead of increase. They still count as saddle node bifurcations.

Bifurcation diagram a plot of fixed points / periodic points (on vertical axis) vs. parameter values (on horizontal axis)

Ex quadratic map



Def A one-parameter family of maps $F_\lambda : X \to X$ undergoes a period-doubling bifurcation at the parameter value λ_0 (called the bifurcation value) if there is an open interval $I \subseteq X$ and $\varepsilon > 0$ such that

(1) when $\lambda \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$, F_λ has one fixed point p_λ in I

(2) when $\lambda \in (\lambda_0 - \varepsilon, \lambda_0)$, p_λ is attracting (resp. repelling)
and F_λ has no 2-cycles in I

(2a) when $\lambda = \lambda_0$, p_λ is neutral
and F_λ has no 2-cycles in I

(3) when $\lambda \in (\lambda_0, \lambda_0 + \varepsilon)$, p_λ is repelling. (resp. attracting)
and F_λ has one 2-cycle in I, q_λ^1, q_λ^2,
which is attracting (resp. repelling)

(4) as $\lambda \to \lambda_0$, $q_\lambda^1 \to p_\lambda$ and $q_\lambda^2 \to p_\lambda$.

Example

① The quadratic map Q_c undergoes a period-doubling bifurcation at $c = \frac{-3}{4}$

Notice when $\frac{-3}{4} < c < \frac{1}{4}$, p_- is attracting

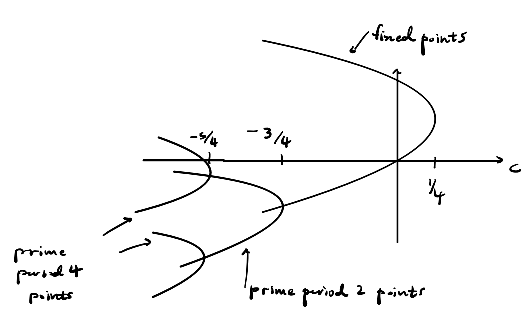
when $c = \frac{-3}{4}$, p_- is neutral $(Q_c'(p_-) = -1)$

and when $-\frac{5}{4} < c < \frac{-3}{4}$, p_- is repelling

and we get an attracting 2-cycle $q_\pm$

② The map Q_c^2 undergoes 2 period-doubling bifurcations at $c = \frac{-5}{4}$. (They occur at q_+ and q_-, where each gives rise to a prime period 4 point for Q_c)

Bifurcation diagram



Other examples Use Desmos/algebra to identify the type of bifurcation that occurs

(saddle node, period doubling, or "other")

① $F_\lambda(x) = x + x^2 + \lambda$, $\lambda = 0$, $\lambda = -1$

② $G_\mu(x) = \mu x + x^3$, $\mu = 1$, $\mu = -1$, $\mu = -2$