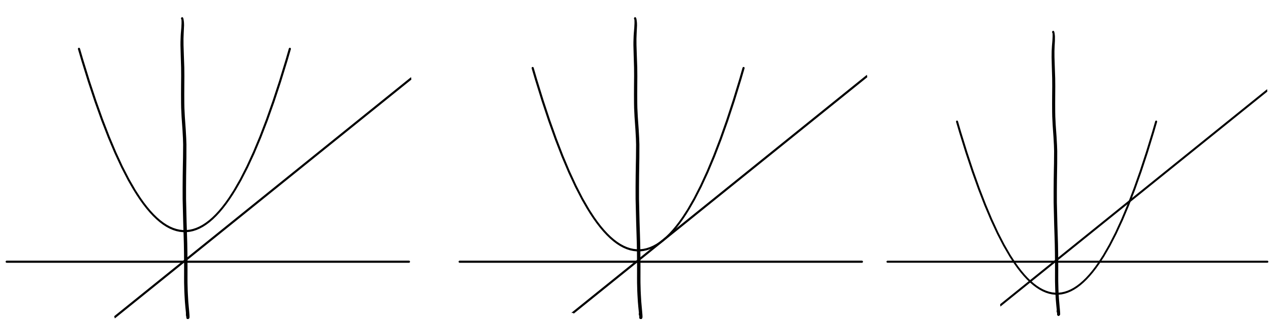
§6.1 Intro to Bifurcations

Our aim is to study the family of functions $Q_c(x) = x^2 + c$ and understand how the behavior of the dynamics changes when we choose different values for c (c is called a parameter of the system).

Main question How do fixed points change when we change c?

Observation!



Notice we go from no fixed points, to one fixed point to 2 as c changes

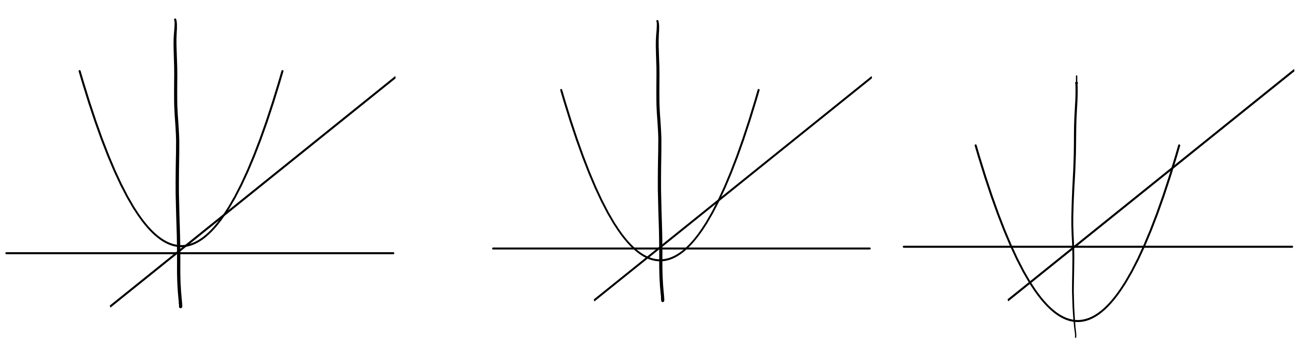
When this transition occurs, we call this a saddle node bifurcation

<u>Question 1</u> Find the range of c values that yield

1. no fixed points
2. one fixed point
3. two fixed points

What are the values of the fixed points?

<u>Observation 2</u> Let's focus on the 2 fixed points range of values.



Q1

$$Q_c(x) = x$$

$$x^2 + c = x$$

$$x^2 - x + c = 0$$

$$x = \frac{1 \pm \sqrt{1-4c}}{2}$$

Notice there are

(1) no solutions when

$$1-4c < 0$$
$$\Rightarrow c > \frac{1}{4}$$

(2) one solution when

$$1-4c = 0$$
$$\Rightarrow c = \frac{1}{4}$$

(3) two solutions when

$$1-4c > 0$$
$$\Rightarrow c < \frac{1}{4}$$

Therefore, when there are two solutions, the fixed points are

$$p_- = \frac{1}{2}\left(1 - \sqrt{1-4c}\right) \quad \text{and}$$

$$p_+ = \frac{1}{2}\left(1 + \sqrt{1-4c}\right).$$

Question 2a What do you notice (graphically) about $Q_c'(x)$ at the fixed points as c changes?

Question 2b Let p_- and p_+ be the left and right fixed points.

1. When is p_+ attracting/repelling/neutral? (which c values)

2. When is p_- attracting/repelling/neutral?

Question 2a

① $Q'(p_+)$ seems to always be greater than 1 (the slope of Q is always greater than the line's slope there)

② $Q'(p_-)$ seems to go from positive to 0 to negative to very negative.

Question 2b

Note $Q'(x) = 2x$

① $Q'(p_+) = 2p_+ = 1 + \sqrt{1-4c} > 1$

for all $c < \frac{1}{4}$

② $Q'(p_-) = 2p_- = 1 - \sqrt{1-4c}$

We must find c values when

$$|Q'(p_-)| < 1 \quad \text{or} \quad = 1 \quad \text{or} \quad > 1$$

Notice $\quad |Q'(p_-)| < 1$

$$\iff \left|1 - \sqrt{1-4c}\right| < 1$$

$$\iff -1 < 1 - \sqrt{1-4c} < 1$$

$$\iff -2 < -\sqrt{1-4c} < 0$$

$$\iff 2 > \sqrt{1-4c} > 0$$

$$\iff 4 > 1-4c > 0$$

$$\iff 3 > -4c > -1$$

$$\iff \frac{-3}{4} < c < \frac{1}{4}$$

So $\quad p_- \quad$ is

- attracting when $\frac{-3}{4} < c < \frac{1}{4}$
- neutral when $c = \frac{-3}{4}$
- repelling when $c < \frac{-3}{4}$.