Fractals - Sierpinski Triangle/Carpet, Topological Dimension and Fractal Dimension

Motivation

This **Fern** consists of many small leaves that branch off a larger one.

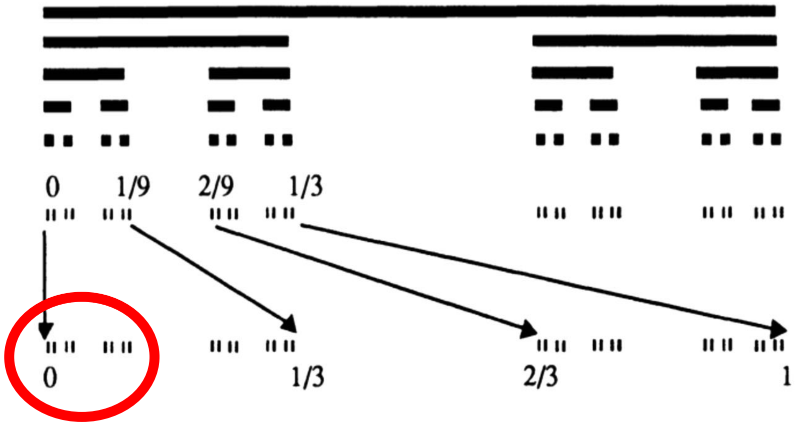
This **Romanesco broccoli** consists of smaller cones spiralling around a larger one.

- Never-ending pattern
- Picture of chaos
- Mathematically, it is **self-similarity**
- *fractal dimension exceeds its topological dimension*

Topological dimension

- Our “everyday” notion of dimension.
- Defined inductively.

- **Definition:** A set S has topological dimension k if each point in S has arbitrarily small neighborhoods whose boundaries meet S in a set of dimension k-1, and k is the smallest non-negative integer for which this is true.
- ***base case:*** A set has topological dimension zero if every point in S can have arbitrarily small neighborhoods whose boundaries do not intersect S



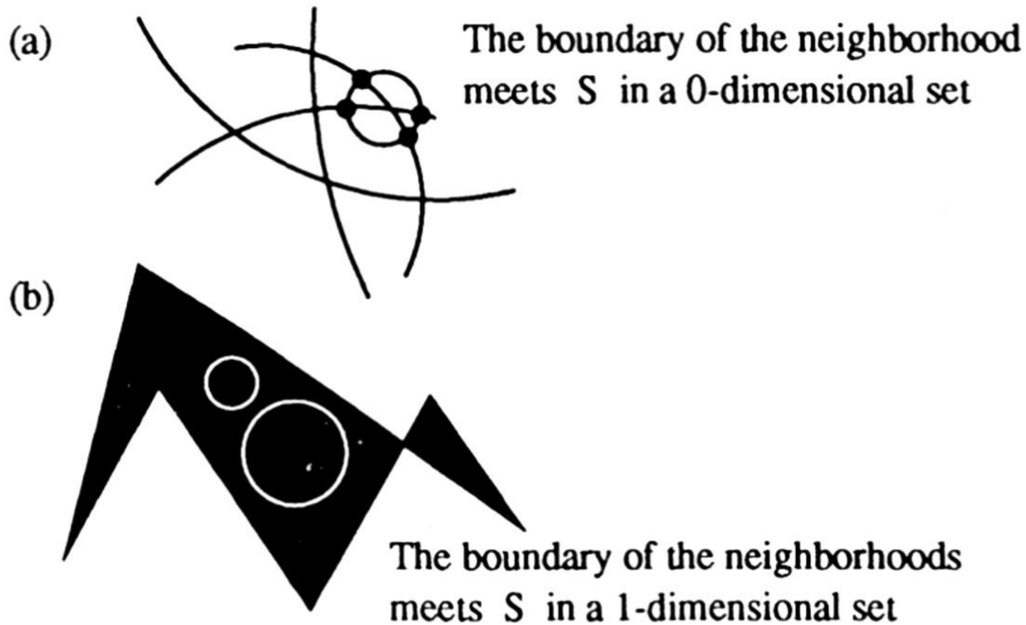


FIGURE 14.13
A set with topological dimension **(a)** 1, **(b)** 2.

Fractal dimension

- **Definition:** A set S is called affine self-similar if S can be subdivided into k congruent subsets, each of which may be magnified by a constant factor M to yield the whole set S
- Fractal dimension is a **more comprehensive measurement of dimension across various types of set S**
- How to calculate fractal dimension?

Definition. Suppose the affine self-similar set S may be subdivided into k pieces, each of which may be magnified by a factor of M to yield the whole set S. Then the *fractal dimension* D of S is

$$D = \frac{\log(k)}{\log(M)} = \frac{\log(\text{number of pieces})}{\log(\text{magnification factor})}.$$

Fractal dimension - Examples

For a line: $D = \frac{log(n^1)}{log(n)} = \frac{1log(n)}{log(n)} = 1.$

For a square: $D = \frac{log(n^2)}{log(n)} = \frac{2log(n)}{log(n)} = 2.$

For a cube: $D = \frac{log(n^3)}{log(n)} = \frac{3log(n)}{log(n)} = 3.$

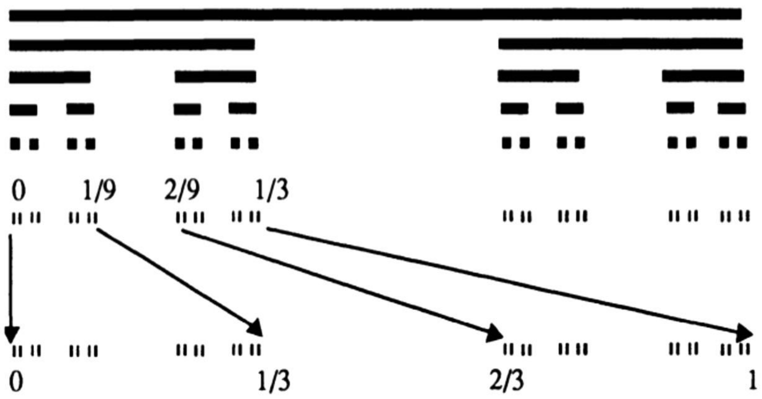
These numbers agree with the topological dimensions, so despite self-similarity, these shapes are not fractals.

What about the fractal dimension of the Cantor middle thirds set? Is it a fractal?
(I.e. is the fractal dimension greater than its topological dimension?

For the **Cantor Middle thirds set**, the number of intervals at each stage of the construction is 2^n and the magnification factor 3^n when applied to any interval, yields the entire Cantor set.

So its fractal dimension

$$D = \frac{\log(\text{Number of pieces})}{\log(\text{magnification factor})} = \frac{\log 2^n}{\log 3^n} = \frac{n \log 2}{n \log 3} = 0.6309\ldots$$



Therefore the Cantor middle thirds set is a fractal because it is affine self similar, and *the fractal dimension exceeds its topological dimension.*

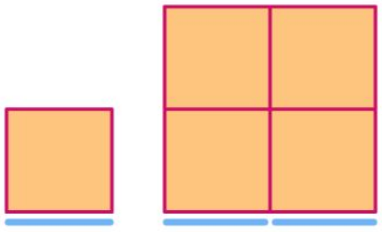
Fractal dimension - Extra examples on how to apply it

Exercise 2: Give explicitly the iterated function system that generates the Cantor middle-fifths set. This set is obtained by the same process that generated the Cantor middle-thirds set, except that the middle fifth of each interval is removed at each stage. What is the fractal dimension of this set?

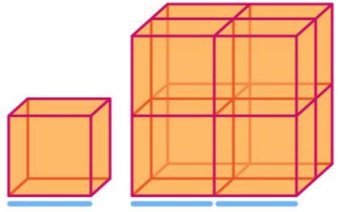
Exercise 3: Consider the set C obtained from the interval [0,1] by first removing the middle-third of the interval and then removing the middle fifths of the two remaining intervals. Now iterate this process, first removing middle thirds, then removing middle fifths. The set C is what remains when this process is repeated infinitely. Is C a fractal? If so, what is its fractal dimension?

Revisiting fractals: a dimension of n will increase its area/volume by 2^n

A line has a dimension of 1. When scaling it by a factor of 2, its length increases by a factor of **2^1 = 2**

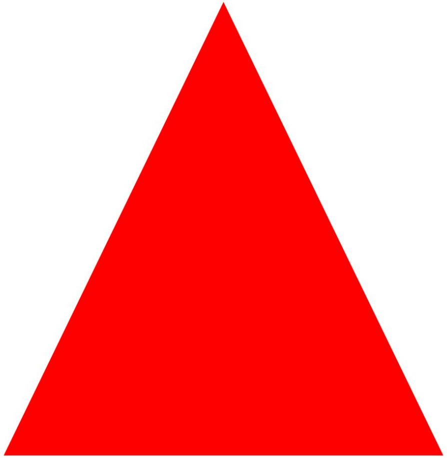


A square has a dimension of 2. When scaling it by a factor of 2, its area increases by a factor of **2^2=4**



A cube has a dimension of 3. When scaling it by a factor of 2, its volume increases by a factor of **2^3=8**

Sierpinski triangle

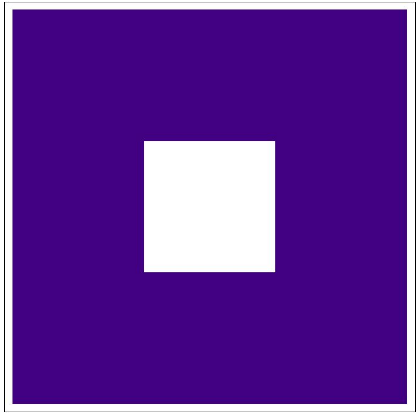


- Begin with the equilateral triangle
- Then, remove from the middle a triangle whose size is exactly **1/2** that of the original triangle
- This leaves 3 smaller equilateral triangles, each of which is exactly **1/2** the size of the original triangle

Calculating its dimension, using the fact that **a dimension of n will increase its area/volume by 2^n:**

$2^d = 3$. In other words, $d = log2(3) = 1.585...$

Sierpinski carpet



- Start with a square and break this square into **9** equal-sized subsquares
- Remove the open middle subsquare
- This leaves **8** equal-sized subsquares

Calculating its dimension, using the fact that **a dimension of n will increase its area/volume by 2^n:**

$$3^d = 8. \; In\ other\ words,\ d = log3(8)$$

How can something has a dimension that is not an integer?

The Sierpinski triangle is something in-between a two-dimensional area, and a one-dimensional line.

In fact, it is **subscribing to the fractal dimension**