Fractals and Iterated Function Systems

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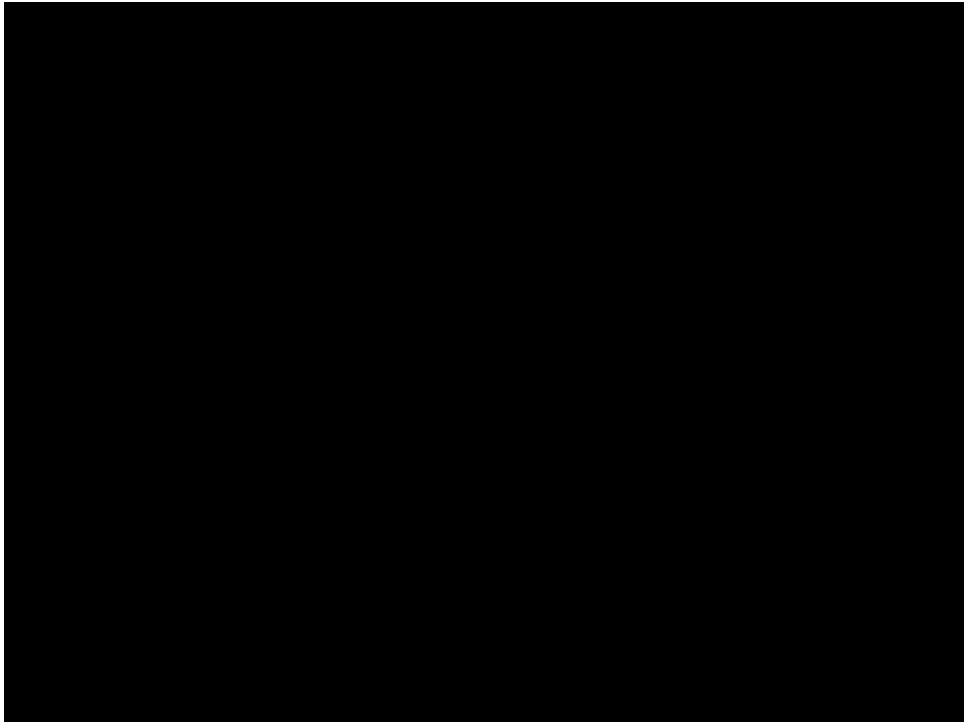
The Chaos Game

1. Draw out three points A, B and C in the plane forming the vertices of a triangle.
2. Choose any point p_0 in the plane as an initial seed
3. Randomly choose A, B or C and place the next point in the orbit halfway to the chosen vertex
4. Repeat Step 3



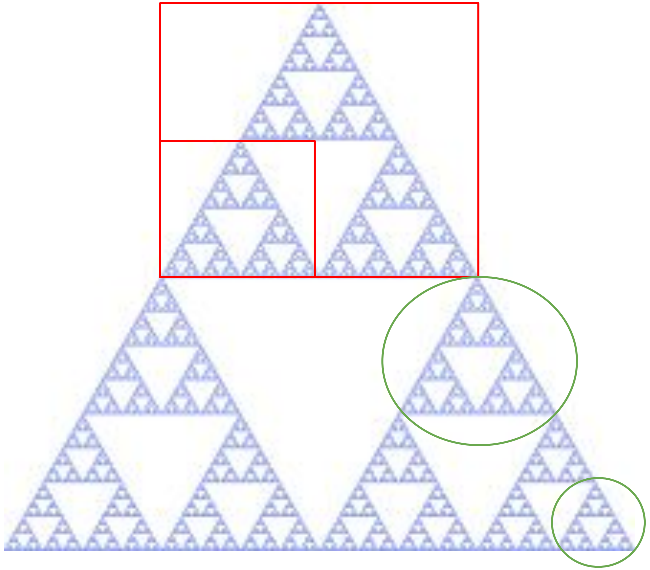
GeoGebra Simulation

Review: Fractals

Definition: A fractal is a subset of R^n which is self-similar and whose fractal dimension exceeds its topological dimension

$$FD = \frac{\log(k)}{\log(M)} = \frac{\log(3)}{\log(2)} = 1.584$$

$$TD = 1$$



Iterated Function Systems: "Intuition"

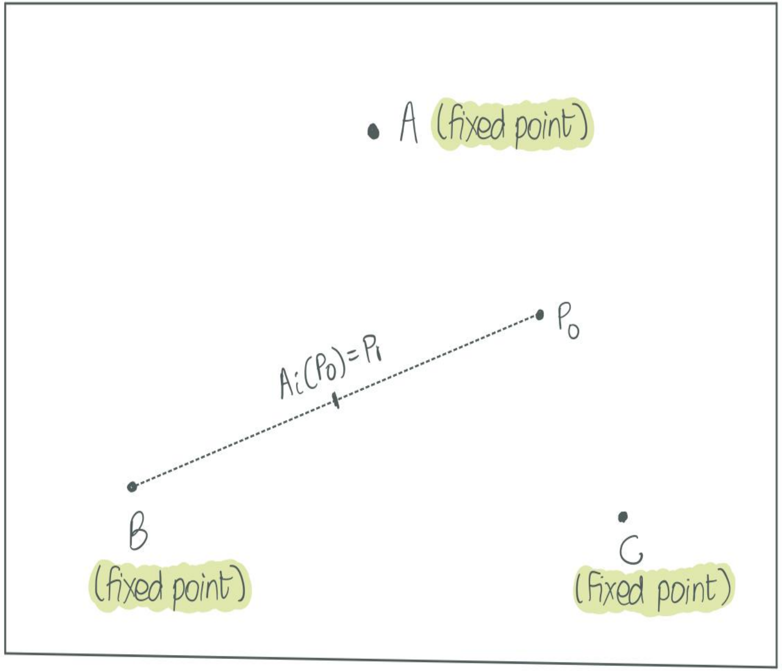
Algebraic way of forming a fractal

Iterated Function Systems: Summary

- Vertices: Fixed Points

- Define a function A_i:
  - A_i "contracts" the distance between p_o and B.

- $$\underbrace{\{\overbrace{A_i}, \ldots, A_n\}}_{\text{Iterated Function System}} (B) - A_i(p_o) = \underbrace{\frac{1}{2}}_{\text{Contraction Factor}} (p_o - B)$$



Iterated Function Systems: Definition

- Comprises of a set of transformations $\{A_1, A_2, \dots, A_n\}$
  - n is the number of fixed points
  - A_i must be a "contracting" function
    - $A_i(p_i) - p = \beta(p_i - p)$
    - $0 < \beta < 1$

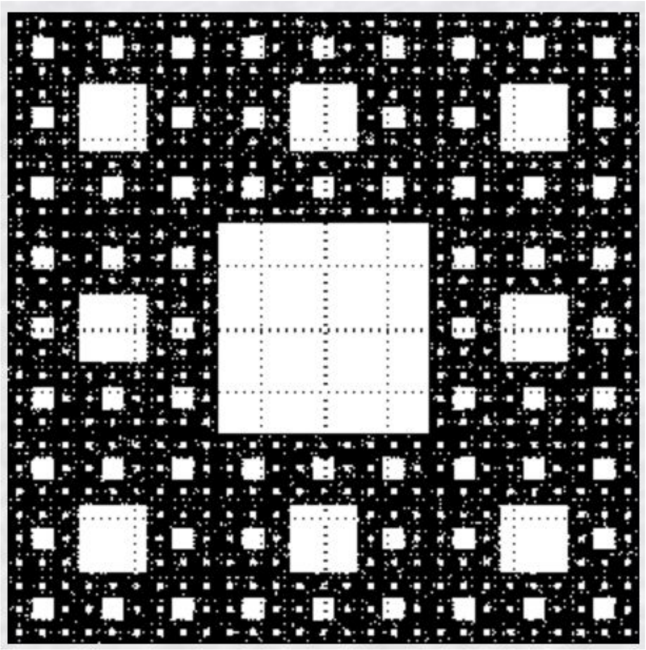
- $$A\begin{pmatrix} x \\ y \end{pmatrix} = \beta \cdot \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} + \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

From IFS to Fractals

- Change in the contraction ratio (β) / initial fixed points } Construct different fractals!

- Examples!

Determining the Iterated Function System of a given Fractal

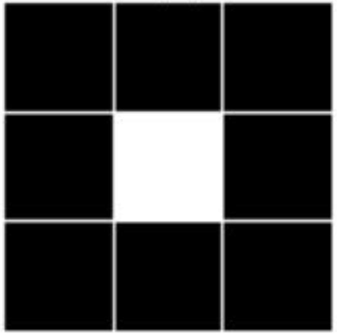


Sierpinski Carpet

Fixed points?
Contraction Factor?

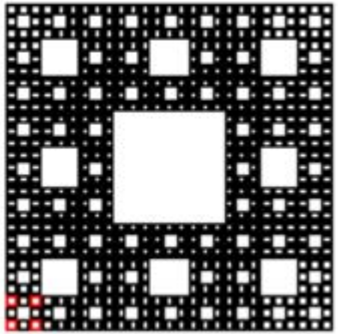
Zeroth
Iteration

First
Iteration



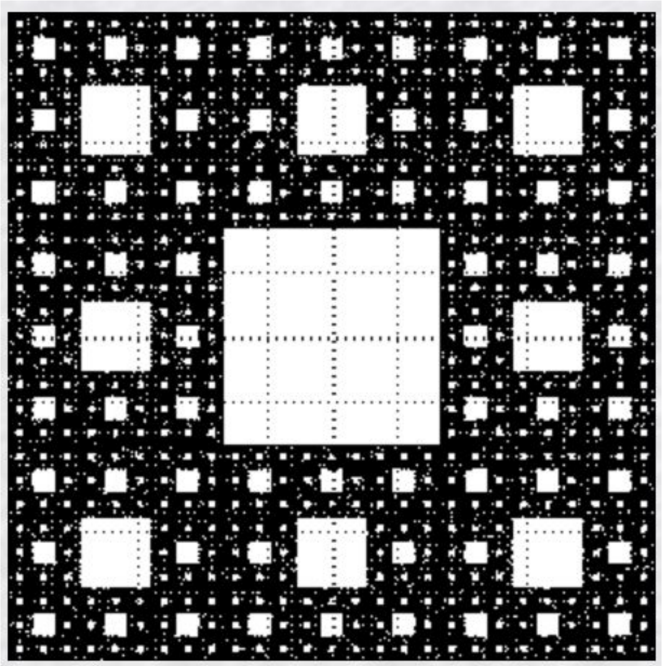
Second
Iteration

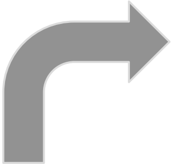
Third
Iteration

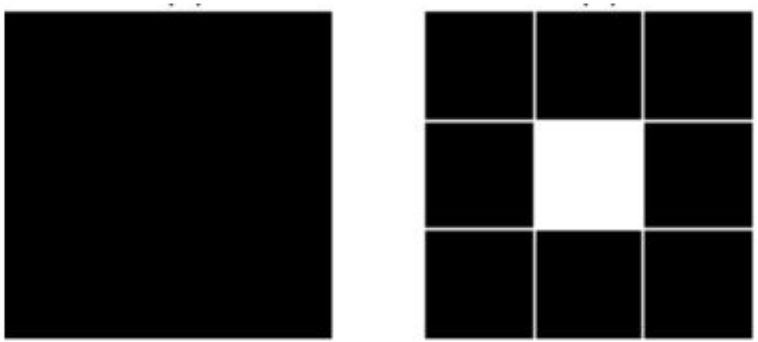


Fourth
Iteration

Sierpinski Carpet





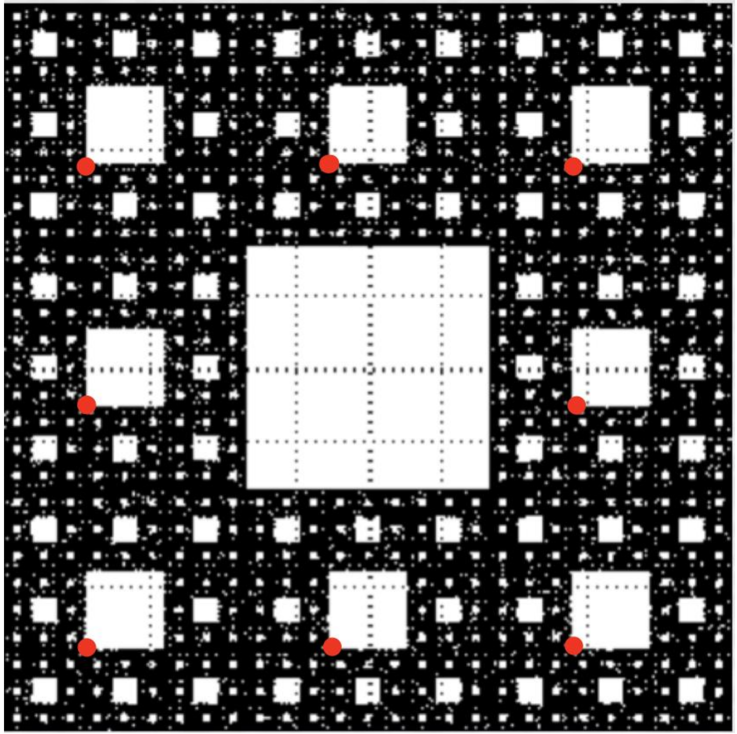


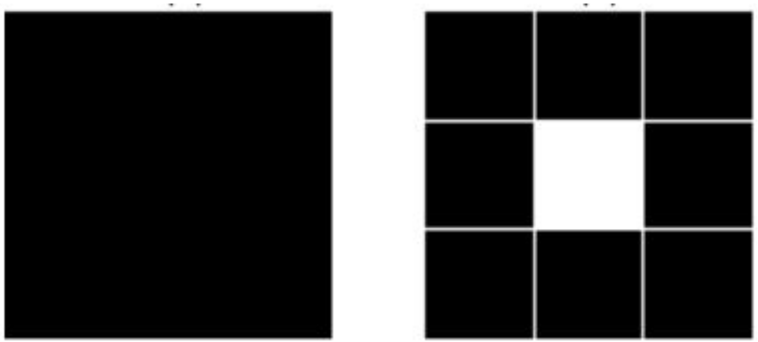
First two iterations

1) Contraction Factor, β = ⅓

2) Number of fixed points = ?

Sierpinski Carpet

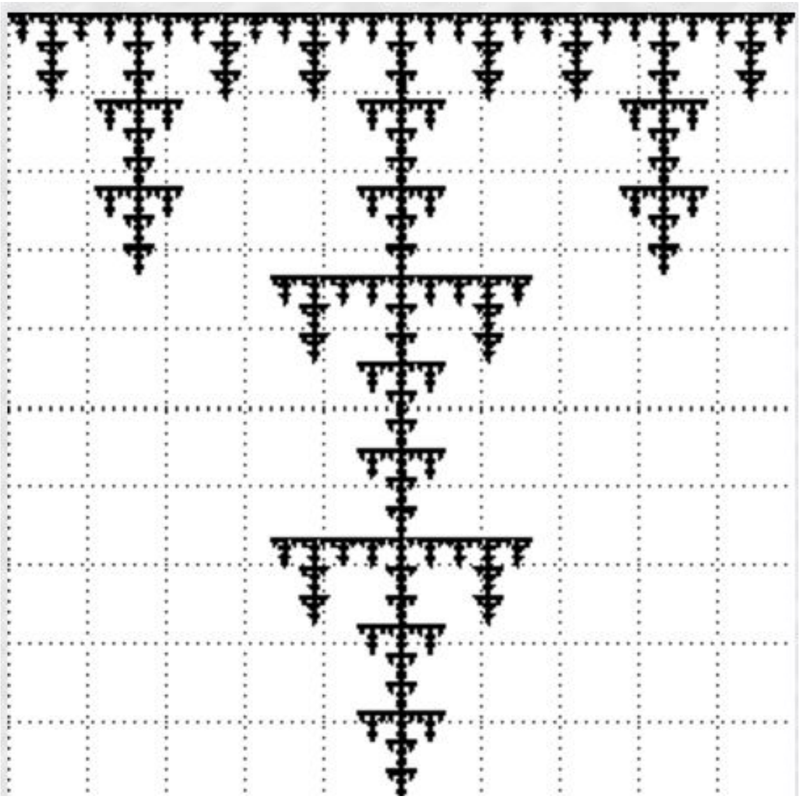




First two iterations

1) Contraction Factor, β = ⅓
2) Number of fixed points = 8
3) $\{A_1, A_2, A_3, \ldots, A_8\}$

Now, to you!



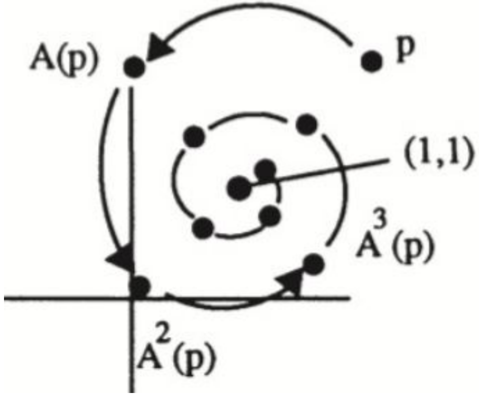
What is β ?

A. ½
B. ⅓
C. ¼
D. ⅕

IFS Examples

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \beta \cdot \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} + \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

Rotation



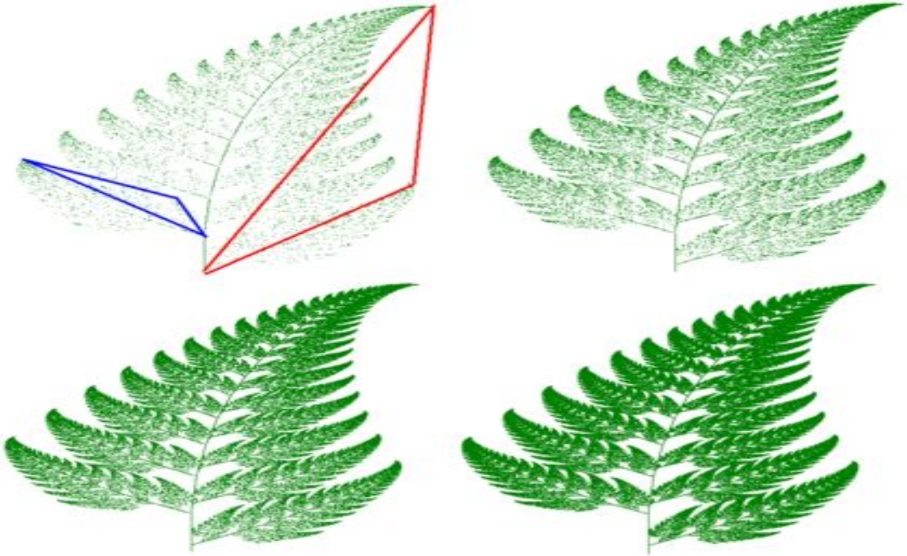
Example. Let $\beta = 0.9$ and $\theta = \pi/2$. Then

$$A\begin{pmatrix} x \\ y \end{pmatrix} = 0.9 \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} x - 1 \\ y - 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

is a linear contraction that fixes

$$p_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Application: Video Games



Barnsley Fern

- Ferns
- Snowflakes
- Carpets
- Any other self similar shape you desire

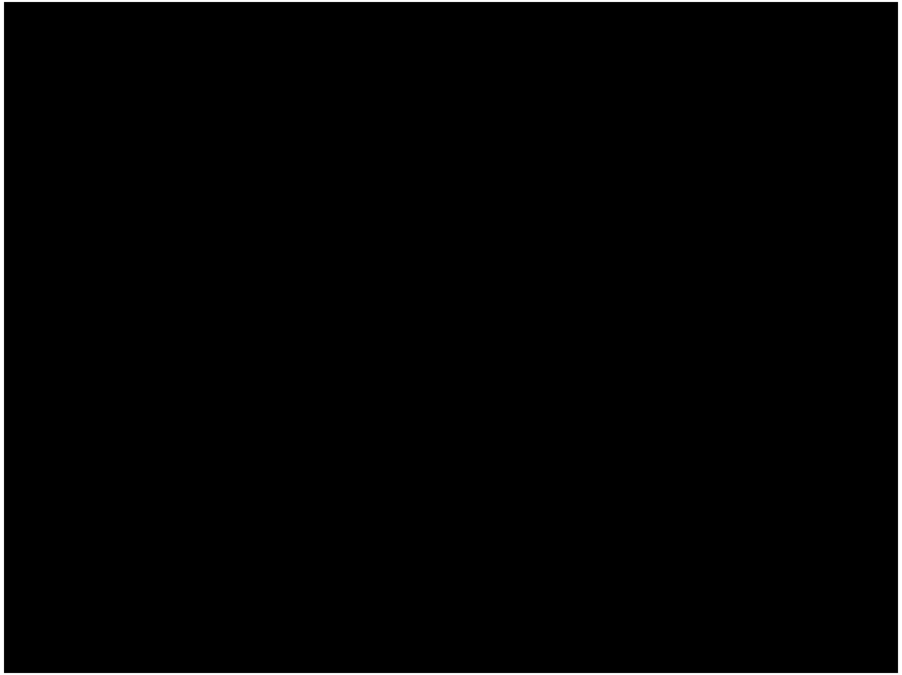
IFS for barnsley ferns

$$\text{A} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0.16 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\text{B} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0.85 & 0.04 \\ -0.04 & 0.85 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 1.6 \end{pmatrix}$$

$$\text{C} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0.20 & -0.26 \\ 0.23 & 0.22 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 1.6 \end{pmatrix}$$

$$\text{D} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -0.15 & 0.28 \\ 0.26 & 0.24 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 0.44 \end{pmatrix}$$



$$A\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0 & 0\\0 & 0.16\end{pmatrix}\cdot\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}0\\0\end{pmatrix}$$

- This function takes any point (x,y) and maps onto a line at the center
- This serves as the stem of the function
- This function is reiterated 1% of the time

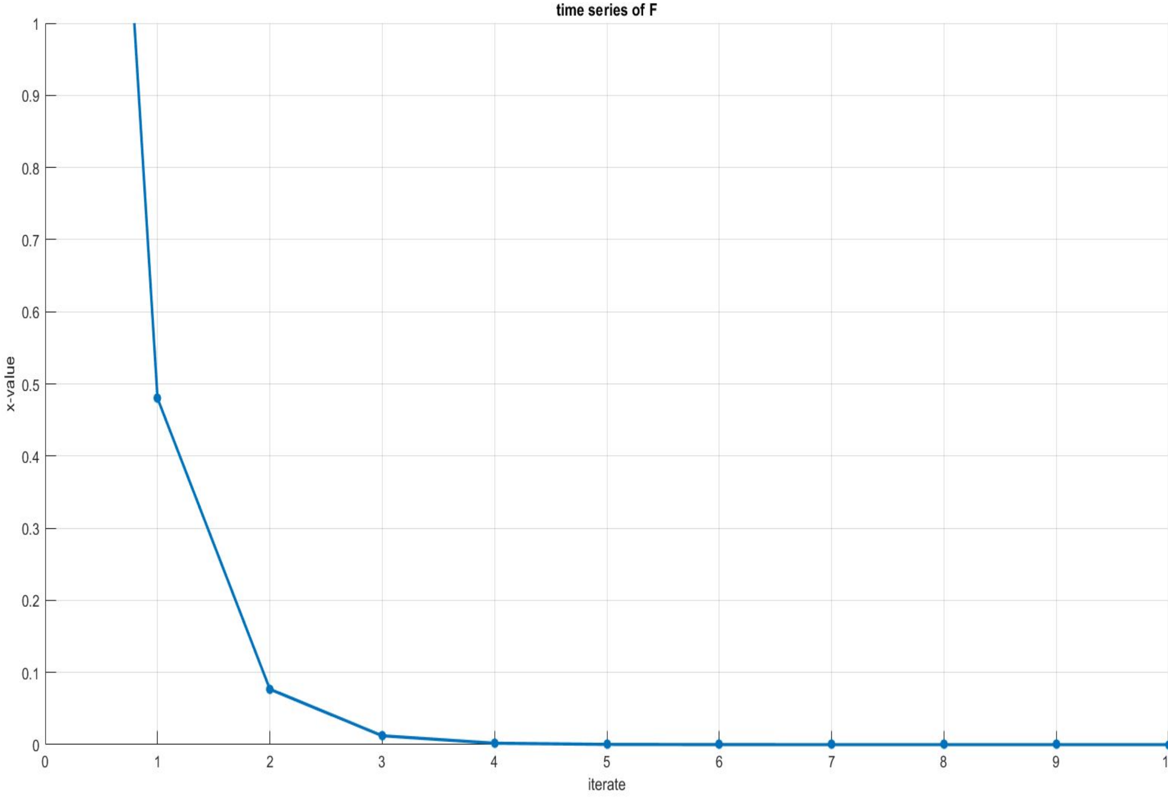
VISUALIZING FUNCTION A

$$A\begin{pmatrix}2\\3\end{pmatrix} = \begin{pmatrix}0 & 0\\0 & 0.16\end{pmatrix}\cdot\begin{pmatrix}2\\3\end{pmatrix} + \begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}0\\0.48\end{pmatrix}$$

$$A\begin{pmatrix}0\\0.48\end{pmatrix} = \begin{pmatrix}0\\0.0768\end{pmatrix}$$

$$A\begin{pmatrix}0\\0.0768\end{pmatrix} = \begin{pmatrix}0\\0.0123\end{pmatrix}$$

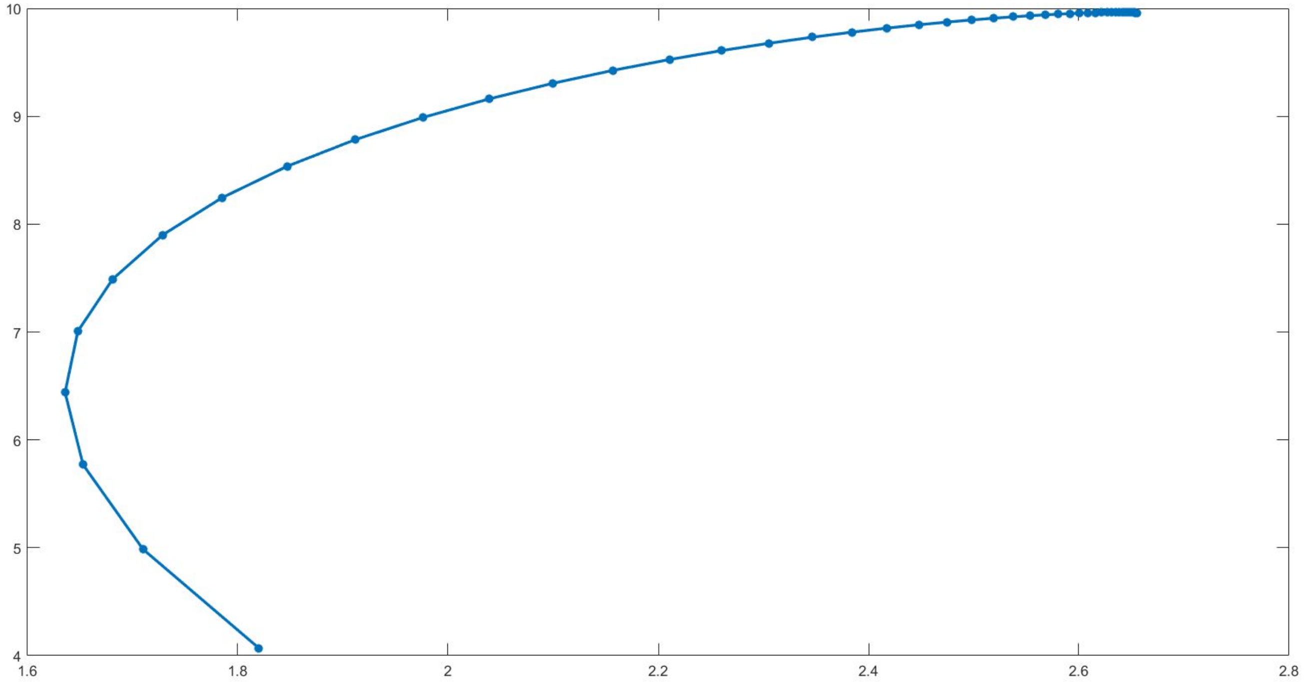
And so on….



$$\mathrm{B}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0.85 & 0.04 \\ -0.04 & 0.85 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 1.6 \end{pmatrix}$$

- This function moves points up and then to the right.
- It results in creating successively smaller leaflets
- This function is reiterated 85% of the time

VISUALIZING FUNCTION B



- Rotation and translation of points evident
- Points moved upwards to form subsequent leaflets
- MATLAB code

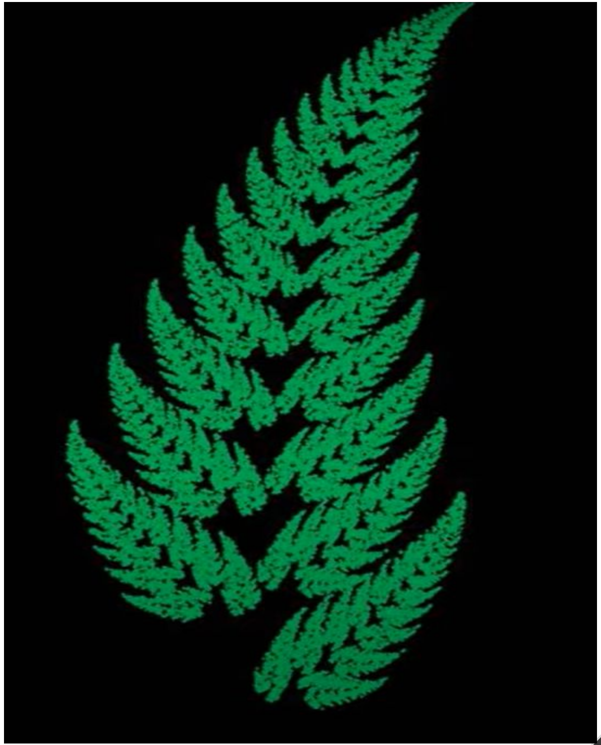
$$\mathrm{C}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0.20 & -0.26\\0.23 & 0.22\end{pmatrix}\cdot\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}0\\1.6\end{pmatrix}$$

- This function rotates points to the left
- It is responsible for the left hand leaflets
- This function is reiterated 7% of the time

$$\mathrm{D}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -0.15 & 0.28 \\ 0.26 & 0.24 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 0.44 \end{pmatrix}$$

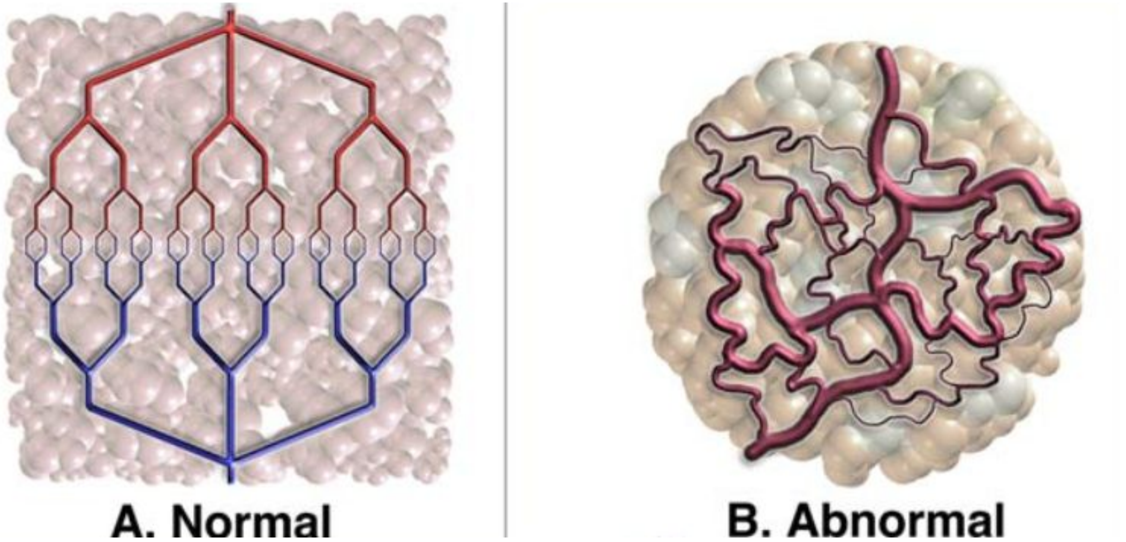
- This function flips points to the right
- It is responsible for creating the large leaflet on the right
- This function is reiterated 7% of the time

Functions are interdependent



Another cool application

- Disease Diagnosis



Questions?

References

- A First Course in Chaotic Dynamical Systems by Robert L. Devaney, ISBN: 9780429280665
- “Chaos Game” (https://www.geogebra.org/m/yr2XXPms)
- Dynamics, Chaos, and Fractals (part 4): Fractals by Evan Dummit, 2015, v. 1.00
- “Barnsley Fern explained” (https://www.youtube.com/watch?v=xoXe0AljUMA&ab_channel=LeiosLabs)
- “A Fractal Journey into the Infinite: Barnsley Fern” (https://www.youtube.com/watch?v=IqQ0DiAgDhc&t=306s&ab_channel=GeorgeRoberts)
- Self written MATLAB code for time series for function B
- Prof. Bob Devaney’s course: Chaotic Dynamical System Lab 5 (http://math.bu.edu/people/bob/MA471/lab5.html)
- Slides 9-13 Images from Prof. Larry Riddle’s IFS webpage, Agnes Scott College (https://larryriddle.agnesscott.org/ifs/carpet/carpet.html)
- Slide 14 Matlab Code from Prof. Tim Chumley’s Dynamical Systems webpage (http://tchumley.mtholyoke.edu/m241/)