Complex Number and Julia Set

By Angelie and Kuma

Complex Number

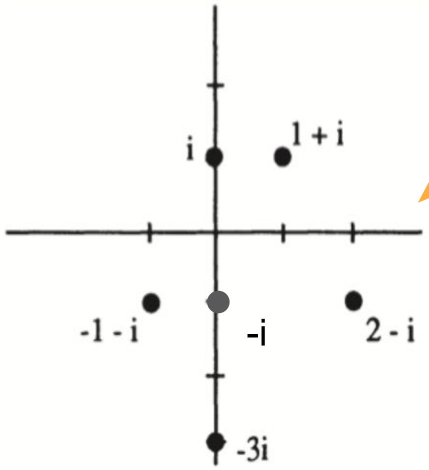
A complex number is of the form $x + iy$, where both x and y are real numbers and i represents the imaginary number, which satisfies $i^2 = -1$. That is, i is, by definition, $\sqrt{-1}$.

If $z = x + iy$ is a complex number, we call x the *real part* of z and y the *imaginary part*. The real number $\sqrt{x^2 + y^2}$ is called the *modulus* of z.

When dealing with arithmetic and complex numbers, addition and multiplication are computed naturally. For example …

Complex numbers can also be represented geometrically, in the complex plane.
In order to plot the complex number $x+iy$ we plot the point (x,y)

We can see that i is plotted at the point (0,1)
and - i is placed at (0,-1)



The plane with points identified as complex numbers is called a *complex plane.*

Polar representation.

Another way of describing points in the complex plane is the *polar representation* of a complex number.

Given the complex number $z = x+iy,$ Its polar representation is determined by the modulus of z

$$|z| = \sqrt{x^2 + y^2},$$

Polar representation, |z|: modulus z, **is the distance from the origin to z.**

Polar angle is the **angle between the positive x-axis and the ray from 0 to z.**

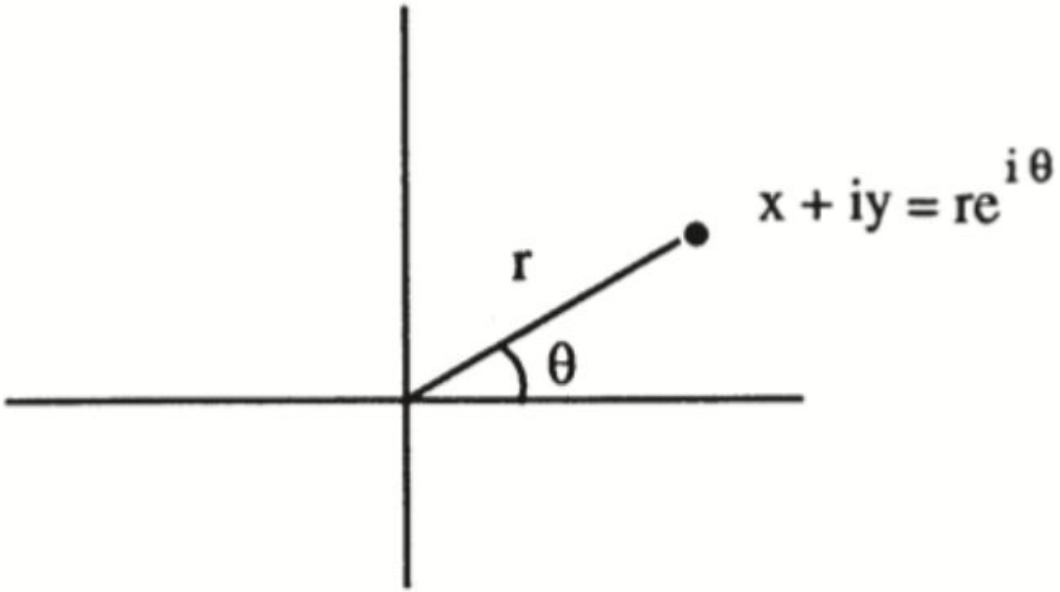


FIGURE 15.2
The polar representation of $z = x + iy$ as $z = r\cos\theta + ir\sin\theta$.

$z = x + iy$ is $z = r\cos\theta + ir\sin\theta$. is our polar representation of z.

Recall Euler's formula from calculus:

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

We thus see that any complex number may also be written in polar form as $z = re^{i\theta}$. For later use, we note that

$$|e^{i\theta}| = \sqrt{\cos^2(\theta) + \sin^2(\theta)} = 1.$$

You may wonder … How does this relate to Julia Sets?

Julia Sets are the place where all of the chaotic behavior of the complex function occurs.

Definitions:

The **filled Julia set** for Q = z^2 + c is the set of all points whose orbits are bounded.

The **Julia set** is the boundary of the filled Julia set.

Def of julia set.

$$Q_c(z) = z^2 + c.$$

Given a fixed c-value, the filled Julia set will consider the fate of all possible initial seeds for the c-value. z_0 can have orbits that go to infinity or seeds which stay bounded.

Those seeds whose orbits **do not escape** (do not tend to infinity) form the ***filled julia set of*** **x^2 + c**. Which Kuma will demonstrate!!

$$z_0 = re^{i\theta}.$$

Polar representation and iteration

$$\begin{aligned} z_0 &= re^{i\theta} \\ z_1 &= r^2 e^{i(2\theta)} \\ z_2 &= r^4 e^{i(4\theta)} \\ &\vdots \\ z_n &= r^{2^n} e^{i(2^n\theta)} \\ &\vdots \end{aligned}$$

This means that there are three possible fates for the orbit of z_0. If $r < 1$, we have

$$r^{2^n} \to 0 \text{ as } n \to \infty.$$

Hence $|Q_0^n(z_0)| \to 0$ as $n \to \infty$. As in the real case, $Q_0(0) = 0$ and $Q_0'(0) = 0$, so 0 is an attracting fixed point. On the other hand, if $r > 1$, then we have

$$r^{2^n} \to \infty \text{ as } n \to \infty,$$

so $|Q_0^n(z_0)| \to \infty$ as $n \to \infty$ in this case.

Computing the Filled Julia Set

N : a maximum number of iterations

-a large number to understand the limiting behavior

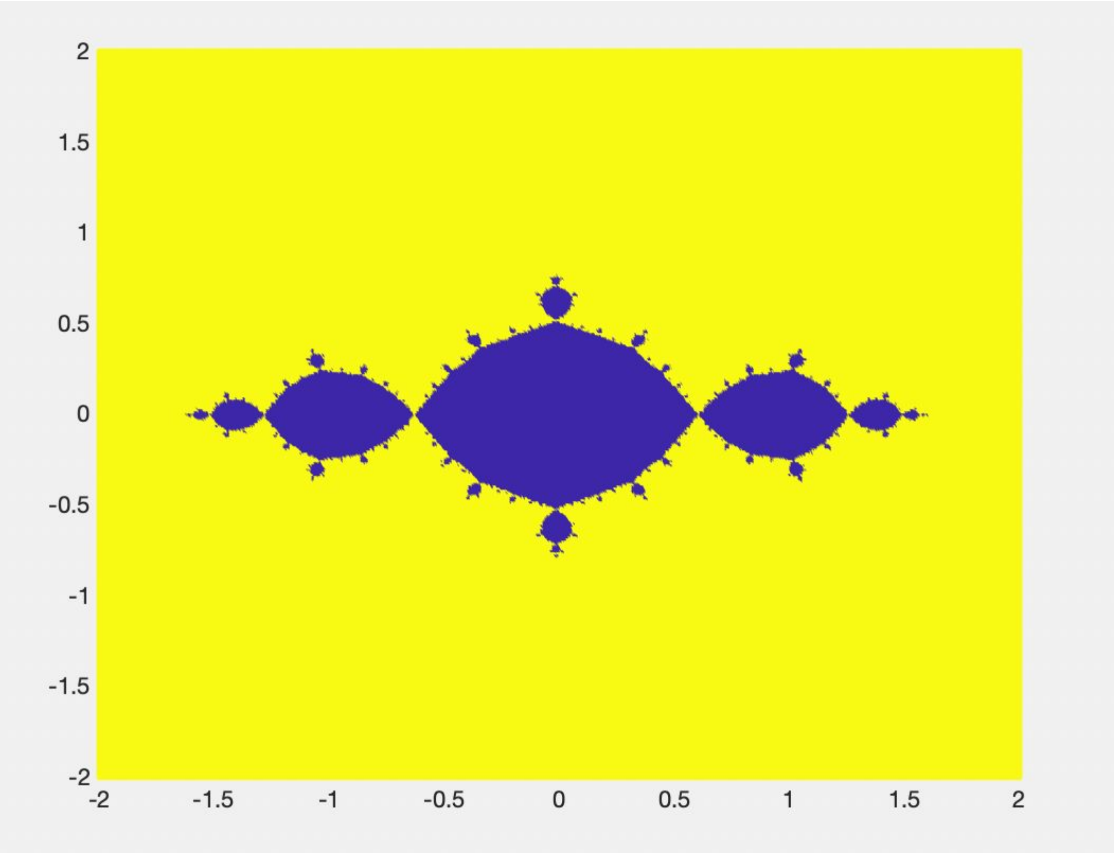
Then, for each point z in the grid, compute the first N points on the orbit of z.

Color yellow:

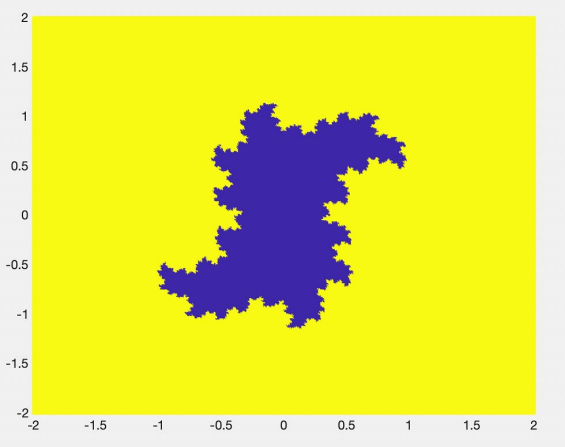
$$|Q_c^i(z)| > \max\{|c|, 2\}\ for\ some\ i \leq N$$

Color blue:

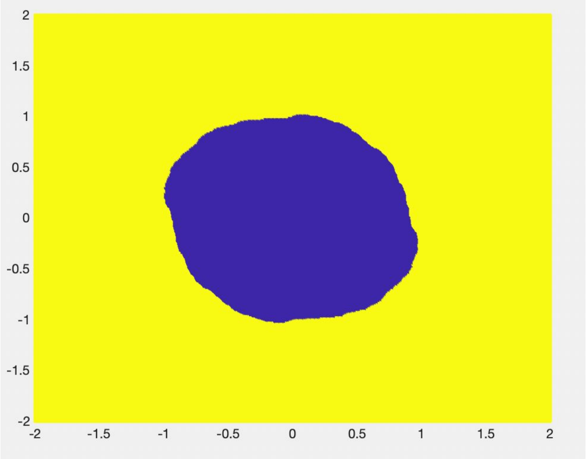
$$|Q_c^i(z)| \leq \max\{|c|, 2\}\ for\ all\ i \leq N$$



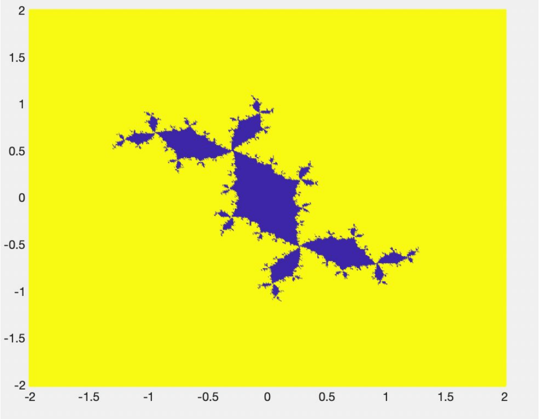
c=-1



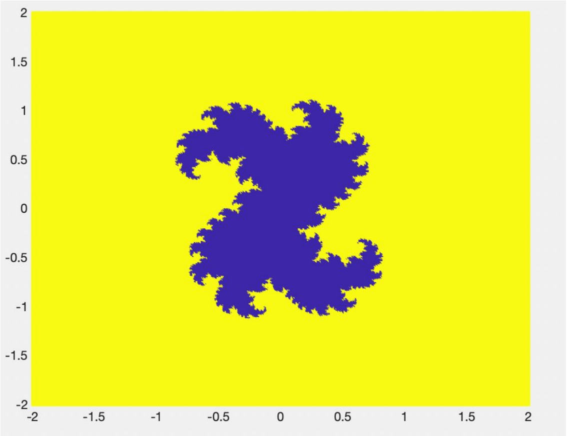
c=0.3-0.4*i*



c = 0.1*i*+0.0546



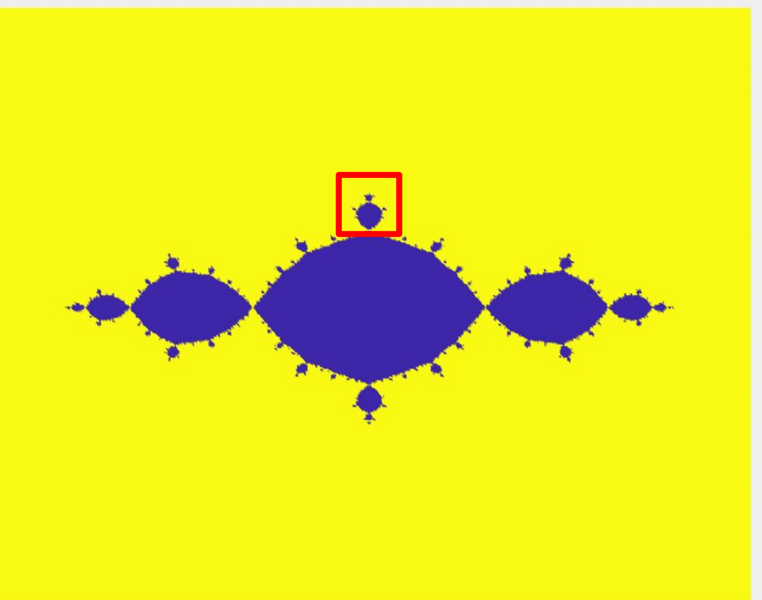
c = -0.1+0.8*i*

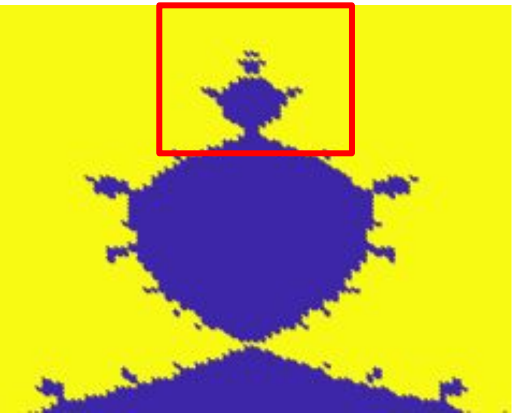


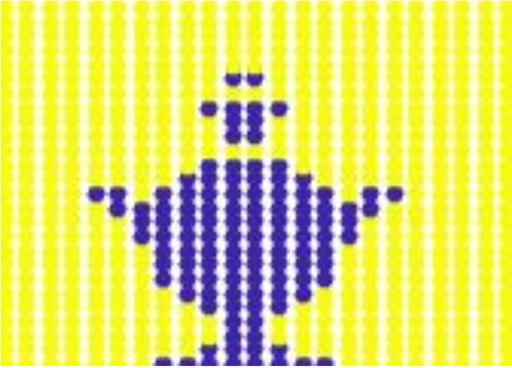
c = 0.34857+0.100089*i*

Observation 1

-For different c-values, K_c assumes a wide variety of shapes. Often, K_c consists of a large connected set in the plane.

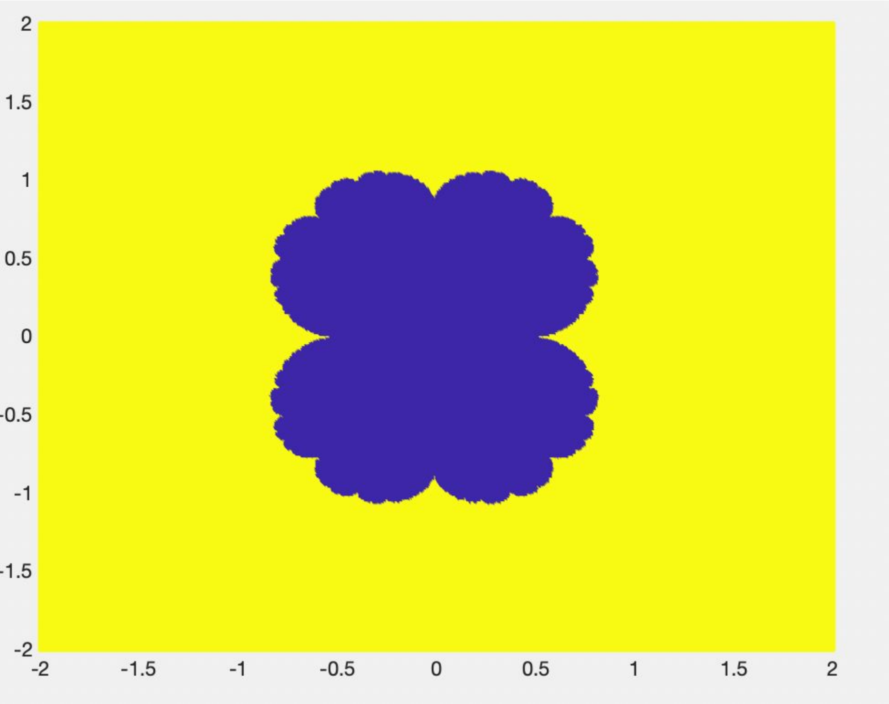




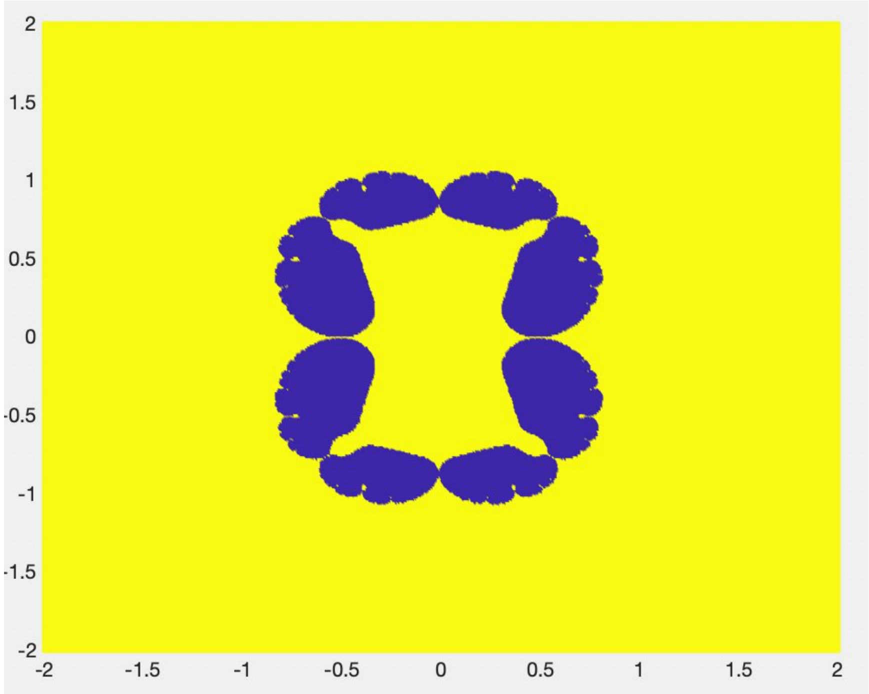


Observation 2

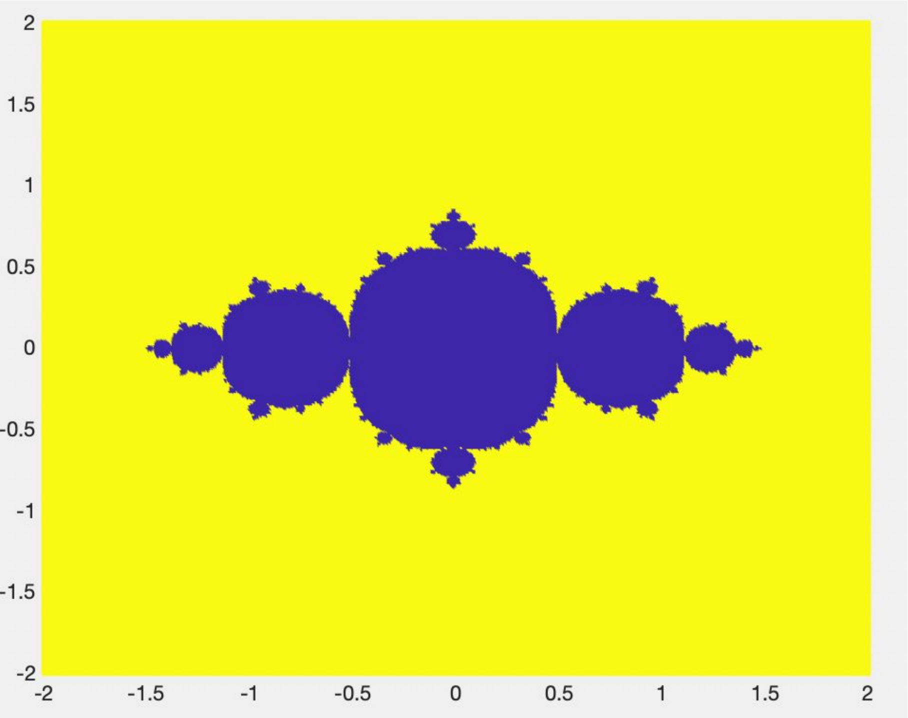
-The Julia sets for Q_c (with c is not 0 or −2) appear to be self-similar sets reminiscent of fractals.


2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2

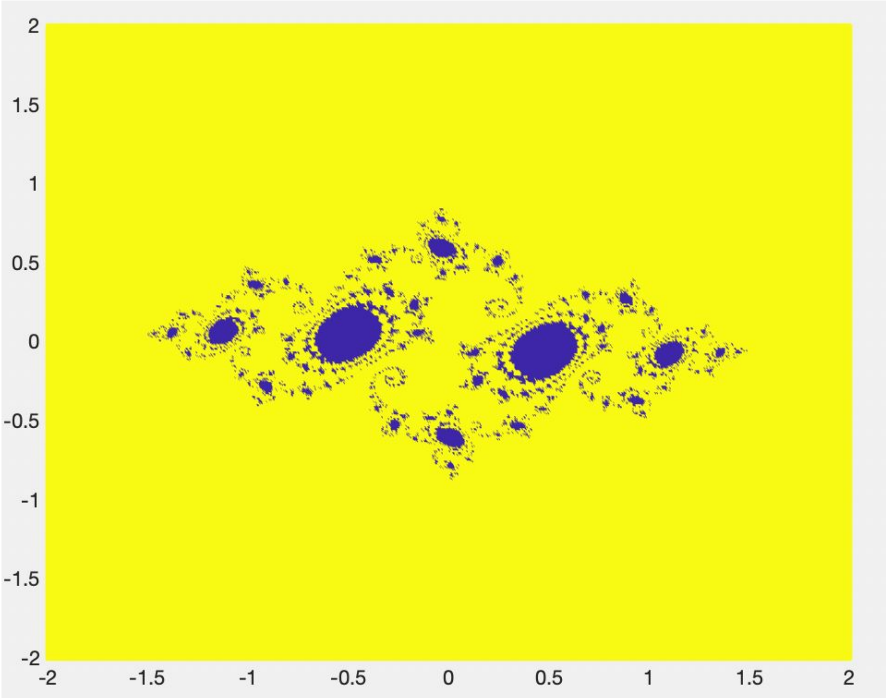
c=0.25


2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2

c=0.251



c=-0.75



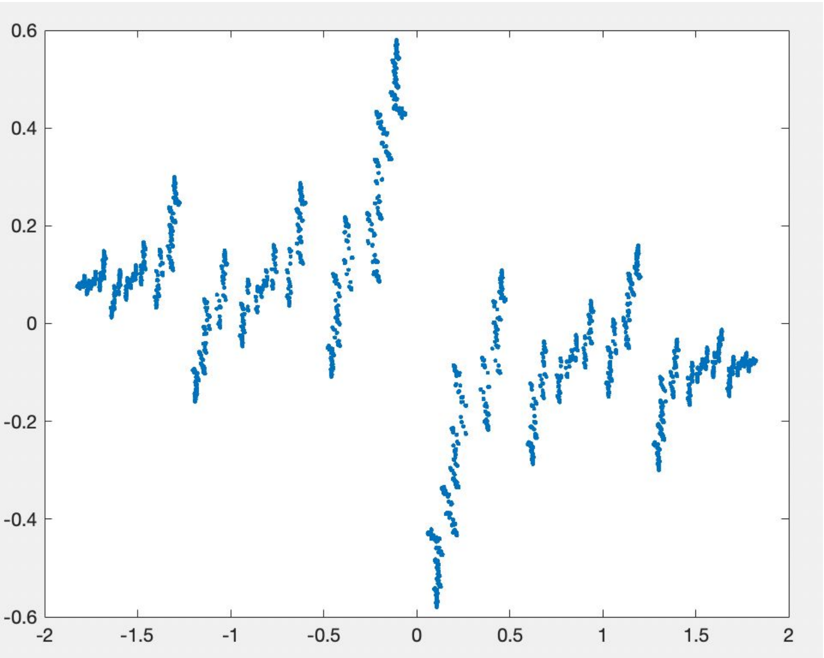
c=-0.75+0.1*i*

Observation 3

-For many c-values, K_c appears to consist of a large number of isolated points and disks. Also, K_c seems to change abruptly from one connected piece to many isolated points and pieces at certain special c-values.

Experiment: Filled Julia Sets and Critical Orbits

-Investigate the relationship between the shape of the filled Julia set of Q_c and the fate of the orbit of 0 under iteration of Q_c.



$c=-1.5 + 0.2i$

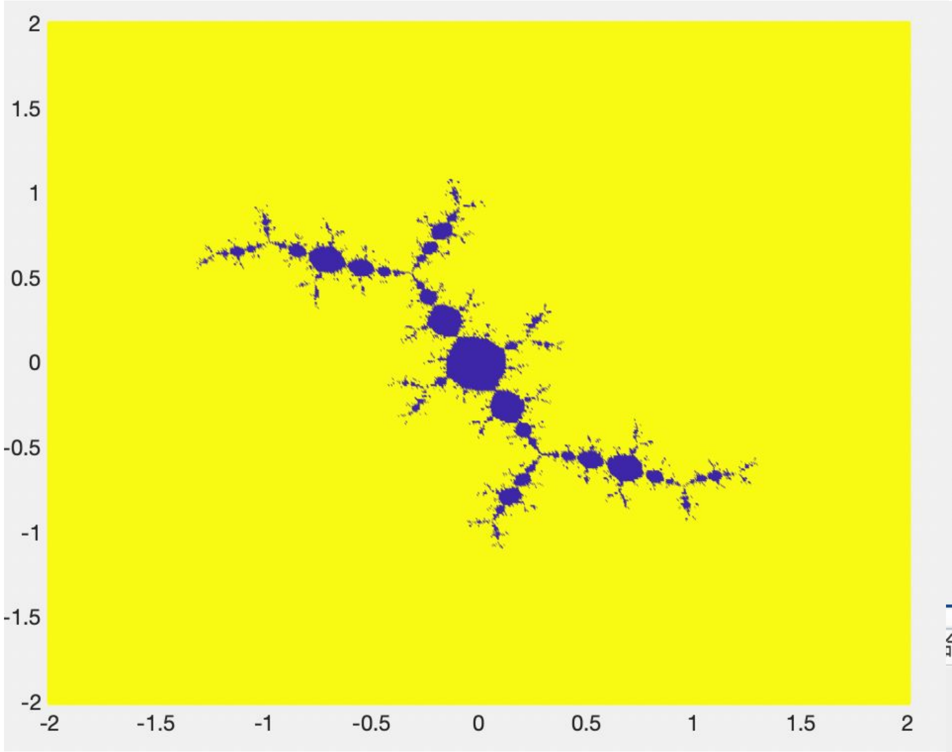
```
%   x0 is the initial point in the orbit
% Outputs:
%   the orbit of F with initial point x0 (printed in the console)
%   this is a list of values of length n+1
%   they are x0, F(x0), F^2(x0), ..., F^n(x0)
c = -1.5 + 0.2i;
F = @(x) x^2+c;
n = 1000;
x0 = 0;
%%%%%%%%%%%%%%%%%%%%%
% Main code (does not need to be modified)
%%%%%%%%%%%%%%%%%%%%%
orbit = zeros(1,n+1);
orbit(1) = x0;
for i = 1:n
    orbit(i+1) = F(orbit(i));
end
% Print out the orbit
orbit(n+1)
```

命令行窗口

```
>> julia_iterator

ans =

  -1.5000 -    Infi
```



c=-0.11 + 0.86*i*

```
%   the orbit of F with initial point x0 (printed in the console)
%   this is a list of values of length n+1
%   they are x0, F(x0), F^2(x0), ..., F^n(x0)
c = -0.11 + 0.86i;
F = @(x) x^2+c;
n = 1000;
x0 = 0;
%%%%%%%%%%%%%%%%%%%%%%
% Main code (does not need to be modified)
%%%%%%%%%%%%%%%%%%%%%%
orbit = zeros(1,n+1);
orbit(1) = x0;
for i = 1:n
    orbit(i+1) = F(orbit(i));
end
% Print out the orbit
orbit(n+1)
```

命令行窗口

```
>> backward_julia
>> julia_iterator

ans =

  -0.1596 + 0.7852i
```

Conclusion

-The orbit of 0 goes to infinity if and only if the Julia set consists of disconnected components.

Thanks for Tim’s codes and help!

Thanks for listening!