Newton's Method

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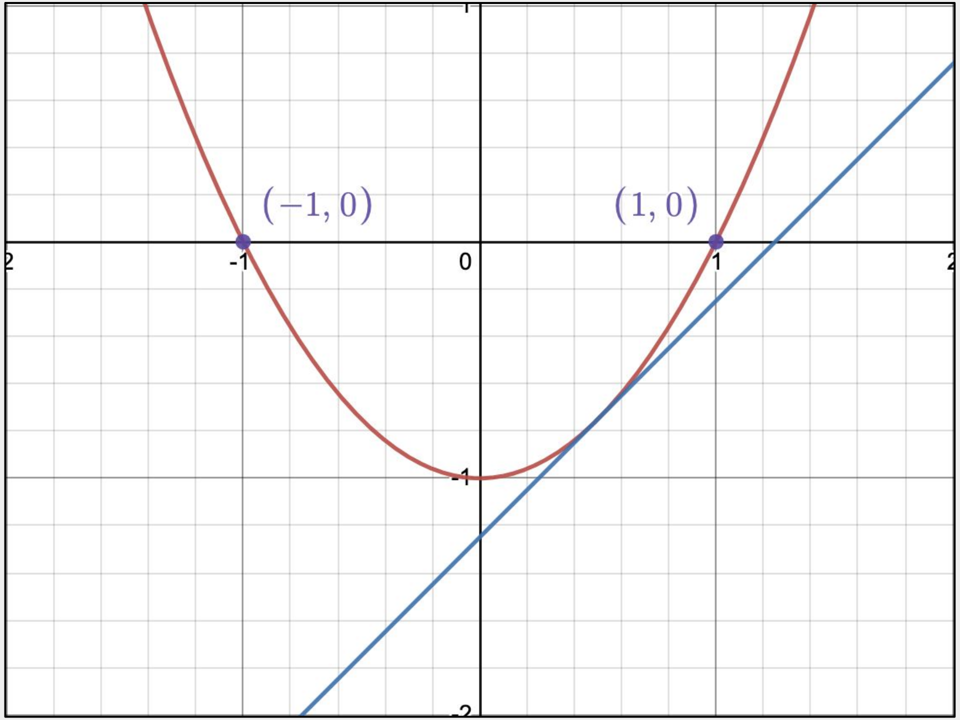
What to Know Before Starting

What is the root of a function?

The *root* of a function is the point or points at which the function crosses the x-axis.



What is a tangent line?

A *tangent line* is a line that touches a curve at one point. The slope of the line matches the curve's slope at that point.

Why Newton's Method?

Newton's method is a tool that will help us to find fixed points and periodic points without having to do as much algebra.

Typically we found fixed points by using

$$F(x) = x$$

Setting it equal to 0 and solving

Notice:

$$F(x) - x = 0$$

This is now a matter of finding the root!

Finding Real Roots of a Function

- Traditionally, we find the root(s) of a function by factoring or applying the quadratic method
  - Ex. $F(x) = x^2 - 5x + 6 = (x-2)(x-3) \rightarrow$ $F(x) = 0$ at $x = 2$ and $x = 3$
- Consider the function $F(x) = 7x^3 + 3x^2 - 5x + 11$. How do we find the root(s)?
- Instead of factoring, we can apply **Newton's method**

Newton's Method and Tangent Lines 1

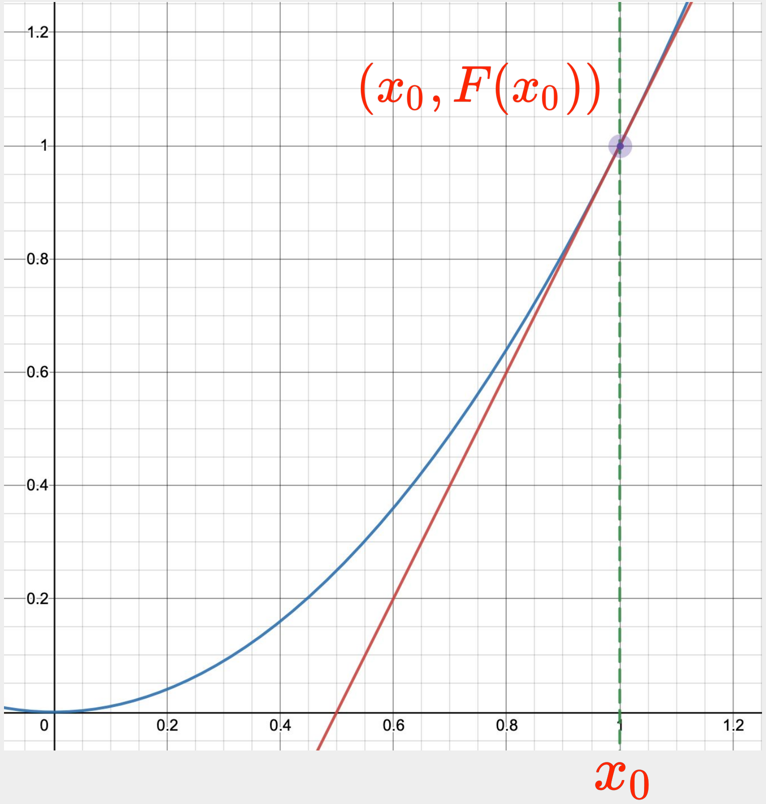
Assume the solution to $F(x) = 0$ is x_0 such that $F(x_0) = 0$

The equation of the line tangent to the graph of $F(x)$ at x_0 is

$$y = F'(x_0)x + B$$

Note that $y = F(x_0)$, so $B = F(x_0) - F'(x_0)x_0$

So $y = F'(x_0)(x - x_0) + F(x_0)$



Newton's Method and Tangent Lines 2

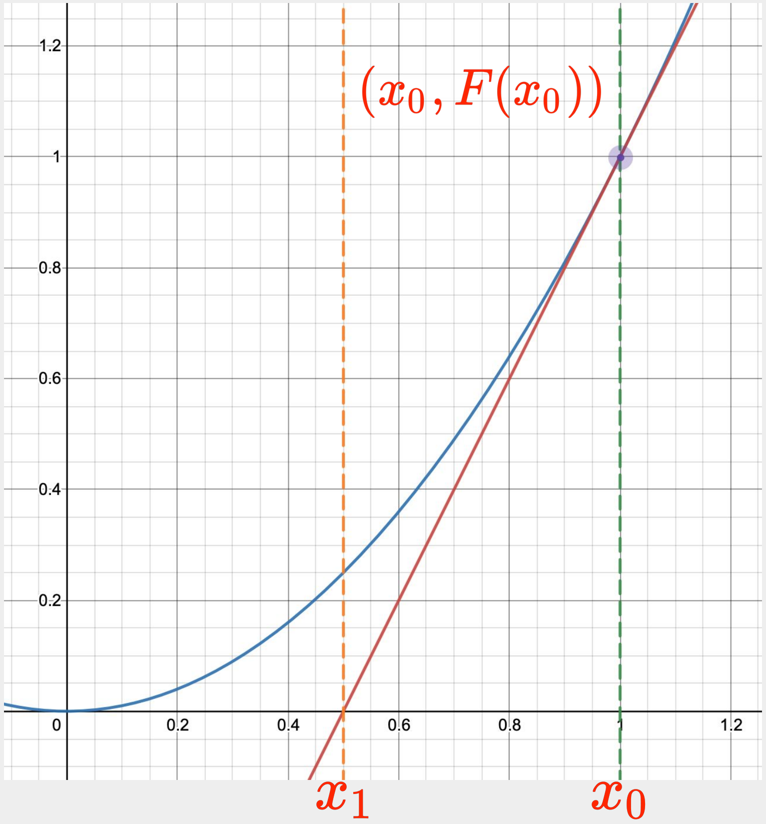
Recall that we have the function

$$y = F'(x_0)(x - x_0) + F(x_0)$$

If we set $y = 0$ and solve for x, we get

$$x_1 = x_0 - \frac{F(x_0)}{F'(x_0)}$$

The point $(x_1, 0)$ is where the tangent line intersects the x-axis



Newton's Method and Tangent Lines 3

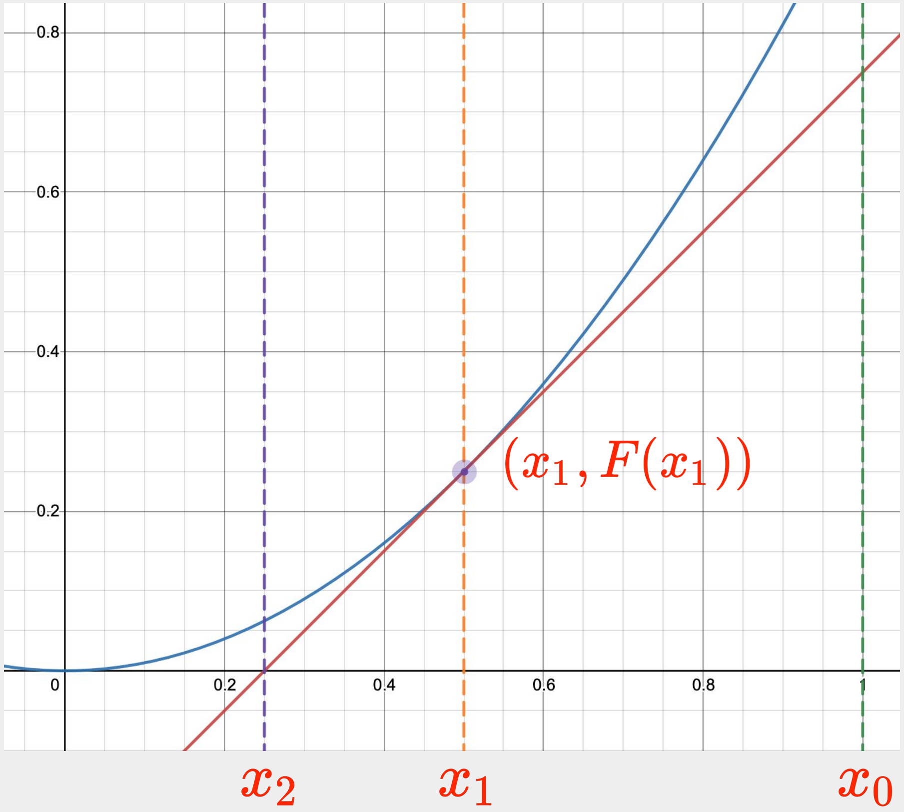
The equation of the line tangent to the graph of $F(x)$ at x_1 is

$$y = F'(x_1)(x - x_1) + F(x_1)$$

If we set y = 0 and solve for x, we get

$$x_2 = x_1 - \frac{F(x_1)}{F'(x_1)}$$

… And so on!



The Newton Iteration Function

In the previous example, we see that each iteration of x gets closer to zero, which is the root of our function

The sequence of points x_0, x_1, x_2,… converges to the root of F in the **Newton iteration function** of F

Definition: Given a function F, the Newton iteration function associated with F is

$$N(x) = x - \frac{F(x)}{F'(x)}$$

Newton's method **succeeds** if the orbit of $N(x_0)$ converges to the root of F. Otherwise, we may have to choose a different initial seed

Newton's Fixed Point Theorem

x_0 is a root of F if and only if x_0 is a fixed point of N. Moreover, x_0 is always an attracting fixed point.

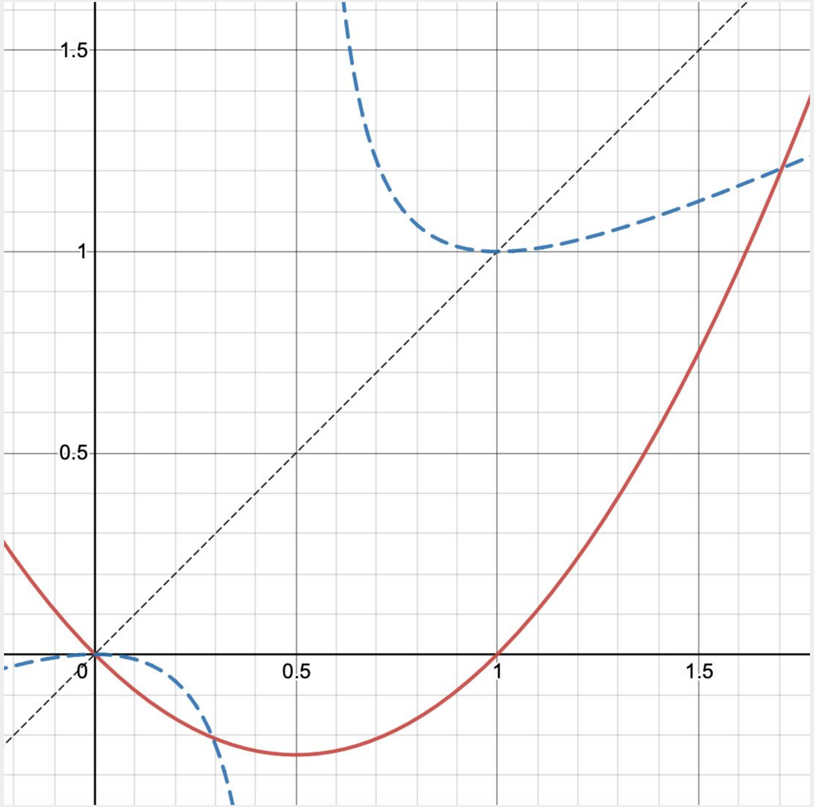
Example:

$$F(x) = x(x-1)$$

$$N(x) = x - \frac{x^2 - x}{2x - 1}$$

$$N(0) = 0 - \frac{0^2 - 0}{2(0) - 1} = 0$$

$$N(1) = 1 - \frac{1^2 - 1}{2(1) - 1} = 1$$



Try it!

1. Find the Newton iteration function $N(x)$ for $F(x) = x^3 - 7x^2 + 8x + 15$

Remember: $N(x) = x - \frac{F(x)}{F'(x)}$

$$N(x) = x - \frac{x^3 - 7x^2 + 8x + 15}{3x^2 - 14x + 8}$$

2. Using $x_0 = 5$, use the Newton iteration function to find the root of F.

$N(5) = 4.61538461538$

$N^2(5) = 4.46085164835$

$N^3(5) = 4.43038864535$

$N^4(5) = 4.43038864535$

$N^5(5) = 4.42917372984$

$N^6(5) = 4.42917372984$

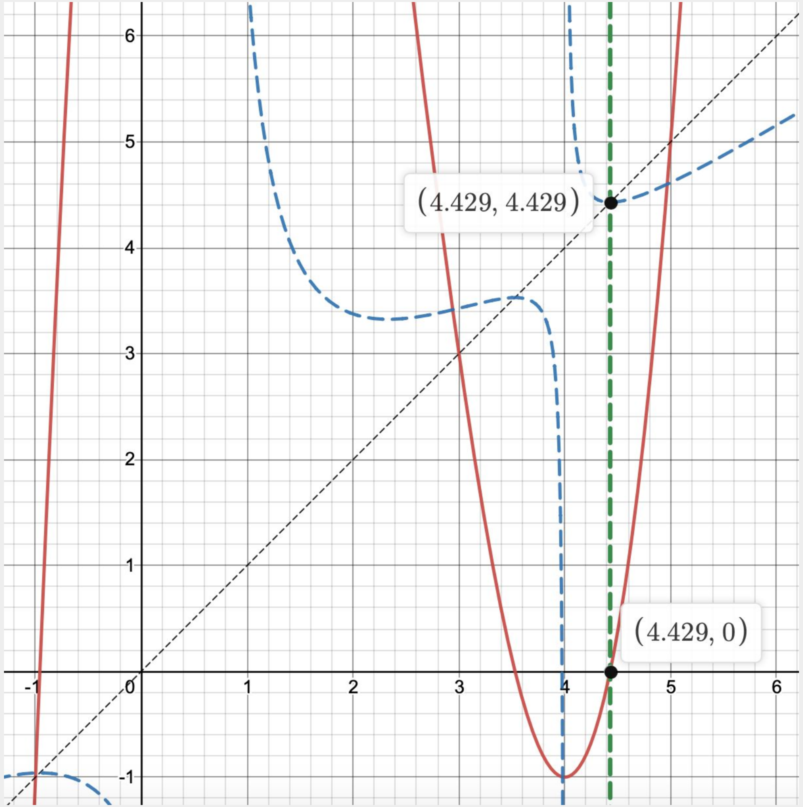
Try it!

Using algebra, we see that the orbit of N(5) converges to 4.429.

Let's check graphically!

$$F(4.429) = 0$$

$$N(4.429) = 4.429$$



Try it! Part 2

1. Find the Newton iteration function $N(x)$ for $F(x) = x^3 - 5x$

$$N(x) = x - \frac{x^3 - 5x}{3x^2 - 5}$$

2. Using $x_0 = 1$, find $N(x_0), N(x_1), N(x_2), N(x_3)$

$$N(x_0) = x - \frac{1^3 - 5(1)}{3(1)^2 - 5} = -1$$

$$N(x_2) = -1$$

$$N(x_1) = x - \frac{-1^3 - 5(-1)}{3(-1)^2 - 5} = 1$$

$$N(x_3) = 1$$

Try it! Part 2

Looking at this graphically we see that x = 1 and x = -1 are period-2 points.

It's important to note that when using Newton's method choosing x_o is imperative to if you find convergence to a fixed point. There is no adhoc way to choose x_o. Here we see the effect of poorly choosing an initial value. Unfortunately we have to discover this through trying examples.