Math 241— How many periodic points are there?

Problem 1. We have seen that it's possible to analytically identify values of c for which the quadratic map $Q_c(x) = x^2 + c$ has bifurcations that give rise to fixed points (a saddle-node bifurcation at $c = 1/4$, a period-doubling bifurcation at $c = -3/4$ where prime period 2 points arise, and another period-doubling bifurcation at $c = -5/4$ where prime period 4 points arise). Moreover, we can look at the orbit diagram to try to numerically detect attracting periodic points at further values of c. Once we get to $c = -2$ the orbit diagram doesn't seem to have any order. Does that mean there are no periodic points for this map? It turns out that's not the case.

a. Find the number of fixed points by plotting the graphs of $y = Q(x) = x^2 - 2$ and $y = x$. Notice all the fixed points are in the interval $[-2, 2]$. What happens to orbits that start outside of this set?

b. Find the number of period 2 points by plotting the graphs of $y = Q^2(x)$ and $y = x$. Notice all the period 2 points are in the interval $[-2, 2]$.

c. Do this again for period 3, period 4, etc. Then make a conjecture for the number of period n points for any $n \geq 1$. Notice all of period 3, period 4, etc. points are in the interval $[-2, 2]$.

d. How many periodic points does Q have? Why don't they appear in the orbit diagram?

Problem 2. Repeat the questions above for the logistic map with $\lambda = 4$:

$$F(x) = 4x(1 - x).$$

Notice this time the action will all take place in the interval $[0, 1]$.

Problem 3. Repeat the questions above for the tent map:

$$T(x) = \begin{cases} 2x & x \leq 1/2 \\ 2 - 2x & x > 1/2 \end{cases}$$

Again, the action will all take place in the interval $[0, 1]$.