Math 206 — Exam 1 review guide

Your exam in class on March 11 will contain about 5 problems, some with multiple parts. You should expect to see a question or two asking you to state some definitions or answer some true false questions where it is important to know definitions. You should also expect a few questions in the style similar to worksheets, homework, and quizzes. It will cover material from Homework 0 to Homework 5. In the textbook, this is material spanning Chapters 2-6. I have outlined some important definitions, theorems, and general topics below. Also, the problems below give you a sampling of some problems like those that will appear on the exam, but it's not necessarily comprehensive, so make sure to review old homework, quizzes, worksheets, and lecture notes.

Definitions

While not necessarily comprehensive, here is an absolutely-must-know list of definitions.

- discrete-time dynamical system, phase space, map, seed, orbit of a seed, nth iterate of a seed
- fixed point, periodic point, prime period of a periodic point, n-cycle, eventually fixed point, eventually periodic point
- attracting fixed point, neutral fixed point, repelling fixed point, attracting n-cycle, neutral n-cycle, repelling n-cycle
- weakly attracting fixed point, weakly repelling fixed point

I won't expect you to memorize the definitions of saddle-node bifurcation and period-doubling bifurcation, but you should be comfortable with the meaning of each piece of these definitions if they are given to you. You should also be comfortable working with the concept of bifurcations in the ways that they came up on Homework 5.

Sample problems

These problems are not comprehensive and there are more than you will see on the exam itself, but they are meant to give you a general idea of the kinds of questions to expect. Also, since you won't have access to Desmos on the exam, I'll provide graphs as appropriate, though you should be comfortable with basic graphing and general calculus concepts like derivatives, slopes, and concavity.

Problem 1. Consider the map $F(x) = x^3 - 2x^2 + x$.

a. Find all fixed points of F and classify each as attracting, repelling, neutral and weakly attracting, neutral and weakly repelling, or neutral and neither weakly attracting nor weakly repelling.

b. Sketch a cobweb diagram showing a few steps of orbits that start near the fixed points you found. *On an exam version of this problem, I would give you a plot with $y = F(x)$ and $y = x$.*

c. Sketch a phase portrait summarizing the behavior of orbits.

d. Express the set $\{x \in \mathbb{R} : \lim_{n\to\infty} F^n(x) = \pm\infty\}$ as an interval or union of intervals.

Problem 2. Consider the 1-parameter family $F_c(x) = x^3 - 2x^2 + cx$.

a. The family undergoes bifurcations at $c = -1, 1, 2$. By looking at Desmos plots, identify each as a saddle-node bifurcation, period-doubling bifurcation, or neither. *On an exam version of this problem, I would show you plots for these and nearby parameter values.*

b. You should find one of the bifurcations is a saddle-node bifurcation. Find an interval I so that for values of c near the bifurcation value, the system goes from having 0 to 1 to 2 fixed points in I.

c. The equation $F_c(x) = x$ has solution $p_1 = 0$. Find the two others; call them p_2, p_3.

d. Find the values of c where p_1 is repelling/attracting/neutral.

e. Sketch the bifurcation diagram for the fixed points using the expressions for p_1, p_2, p_3 you found above. Label each of the three curves in your diagram with the label p_1, p_2, or p_3. No need to use dotted/solid lines to indicate repelling/attracting behavior .

f. What equation would you need to solve in order find prime period 2 points of F_c? Based on your previous work in this problem, how many real-number solutions would this equation have when $c = -1$? What about $c = -1.5$?

Problem 3. Suppose that p is a neutral fixed point of F such that $F'(p) = -1$.

a. Show that p is a fixed point of F^2.

b. Show that p is a neutral fixed point of F^2.

c. The chain rule tells us $(F^2)'(x) = F'(F(x))F'(x)$. Use the product rule to find $(F^2)''(x)$.

d. Compute $(F^2)''(p)$.

Problem 4. State whether each of the following is true or false.

a. If $F(x_0) = x_0$ then $F^3(x_0) = x_0$.

b. If $F(x)$ is a degree 2 polynomial with 2 fixed points, then it is possible for it to have 4 prime period 2 points.

c. If $F : [-1, 1] \to [-1, 1]$ is a continuous function such that $F(-1) = 1$ and $F(1) = -1$, then $F(0) = 0$ and therefore 0 is a fixed point of F.

d. If x_0 is a neutral fixed point such that $F'(x_0) = 1$ and $F''(x_0) = 0$, then there exists an open interval I containing x_0 such that $\lim_{n\to\infty} F^n(x) = x_0$ for all $x \in I$.