Math 241— Bounded orbits for the tent map

In this worksheet, we'll be considering the tent map

$$T(x) = \begin{cases} 3x & x \leq 1/2 \\ 3 - 3x & x > 1/2. \end{cases}$$

and we'll study the set

$$\Gamma = \{x \in \mathbb{R} : T^n(x) \in [0,1] \text{ for all } n \geq 1\}.$$

Problem 1. Use graphical analysis to explain the behavior of any orbit with initial seed $x_0 \notin [0,1]$.

Problem 2. For each integer $n \geq 1$, let $\Gamma_n = \{x \in [0,1] : T^n \in [0,1]\}$. Use Desmos to answer the following.

a. It turns out that $\Gamma_1 = [a,b] \cup [c,d]$. Find a, b, c, d.

b. It turns out that Γ_2 consists of the union of 4 closed intervals. Find them.

c. It turns out that Γ_3 consists of the union of 8 closed intervals. Find them.

d. How many closed intervals does Γ_n consist of? How long is each of them?

Problem 3. How is Γ related to the sets Γ_n defined in the previous problem? The set Γ is called the Cantor Middle-Thirds set. What is meant by this name?

Problem 4. The set Γ is not empty. (In fact it has infinitely many elements!) Find 16 elements in it.

Problem 5 (Bonus, preview for next time)**.** Is $1/4$ an element of Γ? This is a trickier question than you might think. Another way of asking it is whether $1/4$ ever lands in one of the middle-third intervals that gets removed as Γ is constructed. If it does not, then it must be in Γ. In order to answer this, start by finding an infinite sequence $x_1, x_2, x_3, \ldots \in \{0,1,2\}$ such that

$$\frac{1}{4} = \frac{x_1}{3} + \frac{x_2}{3^2} + \frac{x_3}{3^3} + \cdots.$$

The sequence $x_1, x_2, x_3, \ldots$ forms what is called the ternary expansion or ternary representation of $1/4$. What do you think must be true about the ternary expansion of an element of Γ?