Math 301, Fall 2021 — Homework 1

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Due September 10

Instructions. Please submit your solutions to the following problems on Gradescope. Your proof answers should be written in complete sentences and avoid using symbols like $\Rightarrow$, $\therefore$, or $\because$. Edit rough drafts and reread the guidelines for writing mathematics before submitting. Please typeset Problem 1 in LaTeX. For the rest, you may use LaTeX or submit handwritten solutions. When you submit handwritten solutions, make sure your scan is clear, well-aligned, and as readable as possible. Make sure to select which problem is on each page in Gradescope.

Problem 1. Let $S, T \subset \mathbb{R}$ be nonempty, bounded sets with the property $S \subseteq T$.

1. Prove that $\sup S \leq \sup T$.
2. Prove that $\inf T \leq \inf S$.
3. Explain why $\inf T \leq \inf S \leq \sup S \leq \sup T$.

Problem 2. Let $S, T \subset \mathbb{R}$ be nonempty, bounded sets with the property that $s \leq t$ for all $s \in S$ and $t \in T$.

1. Prove that S is bounded above and T is bounded below.
2. Prove that $\sup S \leq \inf T$.
3. Give an example where $\sup S = \inf T$ and $S \cap T = \emptyset$.

Problem 3. For each set below, please give

- Two upper bounds and two lower bounds, if they exist,
- the maximum and minimum, if either exists,
- the supremum and infimum.

1. $\{r \in \mathbb{R} : r^2 \leq 2\}$
2. $\{r \in \mathbb{Q} : r^3 < 8\}$
3. $\{x^2 : x \in \mathbb{R}\}$
4. $\bigcap_{n=1}^{\infty} (1 - 1/n, 1 + 1/n)$

Problem 4. For each description below, give an example of a set with the desired property or explain why no such set exists.

1. A set S with $\inf S \geq \sup S$.
2. A set that contains its infimum but not its supremum.

3. A bounded subset of $\mathbb{Q}$ that contains its supremum but not its infimum.

Problem 5. Let $S \subset \mathbb{R}$ be a nonempty set that is bounded above and let $\alpha = \sup S$.

1. Let $\epsilon > 0$ be a given positive number. Explain why $\alpha - \epsilon$ is not an upper bound for S.
2. Explain why for each $\epsilon > 0$ there exists a corresponding $a \in S$ such that $\alpha - \epsilon < a \leq \alpha$.
3. Explain why there exists an element $a_1 \in S$ such that $\alpha - 1 < a_1 \leq \alpha$.
4. Explain why there exists an infinite list (ie. a sequence) of elements $a_1, a_2, a_3, \ldots \in S$ such that $\alpha - 1/n < a_n \leq \alpha$ for each $n \geq 1$.
5. What do you think you can conclude about the sequence that you've constructed?

Problem 6. Prove that if $a > 0$, then there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < a < n$.

Remark. You'll use the corollaries to the Archimedean property to do this problem, but be careful to note a subtlety that means you have to do a little bit more than just citing the corollaries.

Problem 7. For the next assignment, I plan to assign one or two problems to be done in small groups. Use this space to let me know if you have any group preferences. I plan to assign the groups, and my priority will be to accommodate group mate preferences while balancing the group sizess.