Math 301, Fall 2021 — Homework 10

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Due November 19

Instructions. Please submit your solutions to the following problems on Gradescope. For the group part, please type your answers in LaTeX and submit the output PDF. You'll only submit one write-up for the whole group. **A small change in group work:** for redos on the group problem, I'll ask you to submit redos individually rather than as a group. For the other problems, you may handwrite solutions or use LaTeX. Make sure to select which problem is on each page in Gradescope.

Group problem

Problem 1. The tentative plan for Wednesday is to work on proving the Fundamental Theorem of Calculus through a worksheet. I'll ask you to write up solutions to the worksheet questions as your group problem.

Solo problems

Problem 2. Use the definition of the derivative (ie. the difference quotient limit definition) to calculate the derivatives of the following functions at the indicated points.

1. $f(x) = x^3$ at $a = 2$
2. $f(x) = x^2 \cos x$ at $a = 0$

Problem 3. Consider the function

$$f(x) = \begin{cases} x^2 \sin(1/x) & x \neq 0 \\ 0 & x = 0. \end{cases}$$

1. Let $a \neq 0$. Use Theorems 28.3 and 28.4 to show that f is differentiable at a and to find a formula for $f'(a)$. You may use without proof the fact that $\sin x$ is differentiable and that its derivative is $\cos x$.
2. Give an ϵ-δ proof to show that f is differentiable at 0 and $f'(0) = 0$.

Problem 4. Consider the function

$$f(x) = \begin{cases} x \sin(1/x) & x \neq 0 \\ 0 & x = 0. \end{cases}$$

This function can be proven to be continuous at $x = 0$ using an ϵ-δ proof. Make a conjecture about whether it's differentiable at 0 and prove your conjecture.

Problem 5. Let $f : \mathbb{R} \to \mathbb{R}$ be a given function. Suppose there exists $\alpha > 1$ such that $|f(x) - f(y)| \leq |x - y|^\alpha$ for all $x, y \in \mathbb{R}$. Show that $f'(x) = 0$ for all $x \in \mathbb{R}$ and conclude that f is a constant.

Problem 6. Use the Mean Value Theorem to prove that $|\cos x - \cos y| \leq |x - y|$ for all $x, y \in \mathbb{R}$.