Math 301, Fall 2021 — Homework 11

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Due Decembr 3

Instructions. Please submit your solutions to the following problems on Gradescope. For the group part, please type your answers in LaTeX and submit the output PDF. You'll only submit one write-up for the whole group. **A small change in group work:** for redos on the group problem, I'll ask you to submit redos individually rather than as a group. For the other problems, you may handwrite solutions or use LaTeX. Make sure to select which problem is on each page in Gradescope.

Group problem

No group problem this week.

Solo problems

Problem 1. Give an ϵ-N proof to show that $f_n(x) = \frac{1+\sin nx}{n^2}$ converges pointwise to $f(x) = 0$ on $\mathbb{R}$. Does f_n converge uniformly to f? Why or why not?

Problem 2. Let $f_n : [0, \infty) \to \mathbb{R}$ be given b $f_n(x) = \frac{x^n}{n+x^n}$.

1. Find the pointwise limit $f(x) = \lim_{n\to\infty} f_n(x)$.
2. Determine whether f_n converges uniformly to f on $[0, 1]$ and justify your assertion. *Hint:* $|f_n(x)| \leq 1/n$ *for all* $x \in [0, 1]$.
3. Determine whether f_n converges uniformly to f on $[0, \infty)$ and justify your assertion.

Problem 3. Let $f_n : \mathbb{R} \to \mathbb{R}$ be given by $f_n(x) = \frac{nx}{1+nx^2}$.

1. Find the pointwise limit $f(x) = \lim_{n\to\infty} f_n(x)$.
2. Determine whether f_n converges uniformly to f on $[0, 1]$ and justify your assertion.
3. Determine whether f_n converges uniformly to f on $[1, \infty)$ and justify your assertion.

Problem 4. Let (f_n) be sequence of functions with domain D and suppose $f_n \to f$ pointwise on D. Let $\alpha_n = \sup\{|f_n(x) - f(x)| : x \in D\}$ for each $n \geq 1$. Prove that if $\lim \alpha_n = 0$, then $f_n \to f$ uniformly on D.

Problem 5. Suppose that (f_n) is a sequence of uniformly continuous functions on (a, b). Moreover, suppose that $f_n \to f$ uniformly on (a, b). Prove that f is uniformly continuous. *Hint: use an* $\epsilon/3$ *argument.*

Problem 6. Let $f_n : \mathbb{R} \to \mathbb{R}$ be given by $f_n(x) = \frac{n+\cos x}{2n+\sin^2 x}$.

1. Find the pointwise limit $f(x) = \lim f_n(x)$.
2. Prove that (f_n) converges uniformly on $\mathbb{R}$.
3. Compute $\lim_{n\to\infty} \int_2^7 f_n(x)\, dx$ without integrating f_n.