Math 301, Fall 2021 — Homework 9

Tim Chumley

Due November 12

Instructions. Please submit your solutions to the following problems on Gradescope. For the group part, please type your answers in LaTeX and submit the output PDF. You'll only submit one write-up for the whole group. **A small change in group work:** for redos on the group problem, I'll ask you to submit redos individually rather than as a group. For the other problems, you may handwrite solutions or use LaTeX. Make sure to select which problem is on each page in Gradescope.

Group problem

Problem 1. Prove that the following functions are differentiable at $x_0 = 0$ by giving ϵ-δ proofs.

1. $f(x) = |x|^3$
2. $f(x) = \begin{cases} x^2 & x \geq 0 \\ 0 & x < 0 \end{cases}$

Solo problems

Problem 2. Theorem 19.4 in our textbook says that if $f : D \to \mathbb{R}$ is uniformly continuous and $(x_n) \subset D$ is a Cauchy sequence, then $(f(x_n))$ is a Cauchy sequence. The book gives a proof of this fact that is very readable and just follows definitions.

1. Use Theorem 19.4 and the Bolzano-Weierstrass Theorem to show that if D is a bounded set and $f : D \to \mathbb{R}$ is uniformly continuous, then f is a bounded function on D. Try proving this by contradiction.
2. Use the previous part to give a short proof that $f : (0, 1) \to \mathbb{R}$ given by $f(x) = 1/x^2$ is not uniformly continuous.

Problem 3. For each of the following functions $f : D \to \mathbb{R}$ determine whether f is uniformly continuous on D. If you believe it is, give an ϵ-δ proof showing so. If you believe it is not uniformly continuous, prove it using the technique of your choosing.

1. $f(x) = 3x + 11$, $D = \mathbb{R}$
2. $f(x) = x^3$, $D = [-2, 1]$
3. $f(x) = 1/x, D = [1/2, \infty)$
4. $f(x) = 1/x^3, D = (0, 1)$

Problem 4. Consider the function

$$f(x) = \begin{cases} x^2 \sin(1/x) & x \neq 0 \\ 0 & x = 0. \end{cases}$$

You may assume without proof that this function is continuous on $\mathbb{R}$. Prove that f is uniformly continuous on any bounded set $D \subseteq \mathbb{R}$.

Problem 5. Consider the function

$$f(x) = \begin{cases} x^2 & x \in \mathbb{Q}, \\ 0 & x \notin \mathbb{Q}. \end{cases}$$

Is f differentiable at $x_0 = 0$? If you believe it is, prove it using an ϵ-δ proof. If you believe it is not, prove it using the technique of your choosing.