- Tim / Prof. / Prof. Chumley (he/him)

- Moodle - announcements

- Webpage (tchumley.mtholyoke.edu/m301)
  - notes, worksheets, homework, syllabus
  - updated daily

- Homework (weekly)
  - Written/LaTeX problems, submitted on Gradescope
  - due Fridays at 5 pm

- Quizzes - Wednesdays (first one is Feb 1)

- Exams - one during semester, one during finals

- Participation - come to class, be a good community member, stay in touch when something goes wrong (eg. illness)

- Office hours (tentative)
  - Mondays 4:00-5:00
  - Wednesdays 4:30-5:30
  - Thursdays 1:00-2:00

  } drop in (Clapp 423), no appointment necessary

Introduction

Focus of class: foundations and techniques for continuous mathematics: in depth study of $\mathbb{R}$, limits, continuity, sequences, functions

Number systems we're familiar with (discrete)

$\mathbb{N} = \{1, 2, 3, \dots\}$ (naturals)

$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ (integers)

$\mathbb{Q} = \left\{ \frac{a}{b} : a, b \in \mathbb{Z} \right\}$ (rationals)

Continuous number systems

$\mathbb{R}$, the real numbers, ⟷

($\mathbb{Q}$ + "everything else")

$\mathbb{C}$, complex numbers (not our focus)

Goal: Give more precise definition of $\mathbb{R}$ and understand why $\mathbb{Q}$ is not "good enough"

Two examples for why $\mathbb{Q}$ might not be good enough

Example (algebraic) The equation $x^2 = 2$ has no rational solution. In other words $\sqrt{2} \notin \mathbb{Q}$.

Proof Assume $\sqrt{2} \in \mathbb{Q}$. Then $\sqrt{2} = \frac{a}{b}$, where $\frac{a}{b}$ is a reduced fraction, ie. a and b have no common factor. Therefore $2 = \frac{a^2}{b^2}$ and so

$$a^2 = 2b^2.$$

This means a^2 is even, which implies a is even (prove this yourself). Therefore $a = 2k$ for some integer k. Now we have

$$b^2 = \frac{a^2}{2} = \frac{4k^2}{2} = 2k^2,$$

which implies b^2 is even and b is even. So a and b both have 2 as a common factor, a contradiction.

Example (analytic) Consider the sequence $a_n = \left(1+\frac{1}{n}\right)^n$, $n \geq 1$.

(1) Are elements of sequence rational?

(2) What about limit of sequence?

(1) $a_1 = \left(1+\frac{1}{1}\right)^1 = 2 \in \mathbb{Q}$, $a_2 = \left(1+\frac{1}{2}\right)^2 = \frac{9}{4} \in \mathbb{Q}$, ...

yes, $a_n \in \mathbb{Q}$ for all n

(2) no! $\lim_{n\to\infty} a_n = e \approx 2.71\ldots$, irrational (proof omitted)

Move to come on $\mathbb{R}$ and a precise definition over the next few days.

For now, induction review.

Goal Prove an infinite collection of statements $S_1, S_2, \ldots$ are true.

Technique (1) Prove S_1 is true (base case).

(2) Assume S_n is true for $n>1$ and prove S_{n+1} must be true as a consequence, ie prove $S_n \Rightarrow S_{n+1}$ (induction step)

Examples

① Prove $1+2+\cdots+n = \frac{n(n+1)}{2}$ for all $n \geq 1$.

The base case, $n=1$, gives $1 = \frac{1(1+1)}{2}$, which is obviously true. Assume the formula holds for a general value $n>1$. Now we'll try to prove it for $n+1$. Observe that

$$1+2+\cdots+n+n+1 = \frac{n(n+1)}{2} + n+1 \quad \text{(by induction hypothesis)}$$

$$= (n+1)\left(\frac{n}{2}+1\right)$$

$$= \frac{n+1}{2} \cdot 2\left(\frac{n}{2}+1\right)$$

$$= \frac{(n+1)(n+2)}{2}.$$

② Prove $n^2 \geq 3n$ for all $n \geq 3$.

The base case, $n=3$, gives $9 \geq 9$, which is obviously true. Assume the inequality holds for a general value $n>1$. Observe that

$$(n+1)^2 = n^2 + 2n + 1$$

$$\geq 3n + 2n + 1 \quad \text{(by induction hypothesis)}$$

$$\geq 3n + 2 + 1 \quad \text{(since } n \geq 1\text{)}$$

$$= 3(n+1).$$