§ 2.7 Properties of Infinite Series, cont'd.

Theorem (Absolute Convergence Test) If the series $\sum_{k=1}^{\infty} |a_k|$ converges, then the series $\sum_{k=1}^{\infty} a_k$ converges as well. (We say $\sum_{k=1}^{\infty} a_k$ converges absolutely in this case. When $\sum_{k=1}^{\infty} |a_k|$ diverges but $\sum_{k=1}^{\infty} a_k$ converges we say $\sum_{k=1}^{\infty} a_k$ converges conditionally.)

Proof To show $\sum_{k=1}^{\infty} a_k$ converges, we can show for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that when $n > m \geq N$, $|a_{m+1} + \cdots + a_n| < \varepsilon$. Let $\varepsilon > 0$. Since $\sum_{k=1}^{\infty} |a_k|$ converges, there exists $N \in \mathbb{N}$ such that $\big| |a_{m+1}| + \cdots + |a_n| \big| < \varepsilon$ when $n > m \geq N$. Suppose $n > m \geq N$. Then

$$|a_{m+1} + \cdots + a_n| \leq |a_{m+1}| + \cdots + |a_n|$$
$$= \big| |a_{m+1}| + \cdots + |a_n| \big| < \varepsilon.$$

Theorem (Alternating Series Test) Suppose (a_n) is a sequence such that

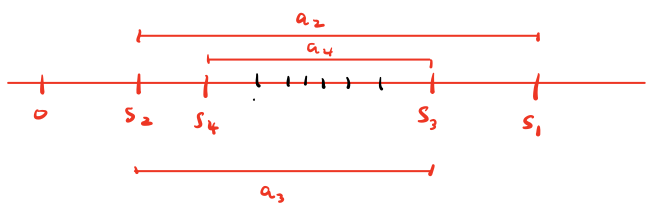
(1) $a_n \geq 0$ for all $n \in \mathbb{N}$

(2) (a_n) is decreasing

(3) $a_n \to 0$.

Then $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges.

Proof Exercise 2.7.1.



Example $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges conditionally

$$S = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \cdots \qquad = \ln(2) \approx 0.69$$

Question What happens if we rearrange the terms?

Is $1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \cdots$

the same as $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$?

No!

$$S = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \tfrac{1}{7} - \tfrac{1}{8}$$

$$+ \quad \tfrac{1}{2} S = \tfrac{1}{2} - \tfrac{1}{4} + \tfrac{1}{6} - \tfrac{1}{8} + \cdots$$

$$\tfrac{3}{2}\ln(2) = \tfrac{3}{2} S = 1 + \tfrac{1}{3} - \tfrac{1}{2} + \tfrac{1}{5} + \tfrac{1}{7} - \tfrac{1}{4} + \cdots$$

Outline of the Proof of the Alt. Series Test

① Show (S_{2k}) and (S_{2k-1}) are bounded and monotone

② By MCT

$$S_{2k} \to A \quad \text{as } k \to \infty$$

$$S_{2k-1} \to B \quad \text{as } k \to \infty$$

for some $A, B \in \mathbb{R}$.

Show $\lim_{k \to \infty} (S_{2k} - S_{2k-1}) = 0$ and use this to explain why $A = B$.

③ Now prove $S_n \to A$ using an $\varepsilon - N$ proof. Construct your N using the N's that you get from (S_{2k}) and (S_{2k-1}) converging. You'll probability find it helpful to have 2 cases at a certain step of your proof.

Def Let $\sum_{k=1}^{\infty} a_k$ be a given series. A series $\sum_{k=1}^{\infty} b_k$ is called a rearrangement of $\sum_{k=1}^{\infty} a_k$ if there exists a bijection $f : \mathbb{N} \to \mathbb{N}$ such that $b_{f(k)} = a_k$ for all $k \in \mathbb{N}$.

Riemann Rearrangement Theorem If a series converges conditionally, for every $A \in \mathbb{R}$, there exists a rearrangement of the series that converges to A.

Theorem If a series converges absolutely, then any rearrangement converges to the same value.

Proof Let $\sum_{k=1}^{\infty} a_k$ be an absolutely convergent series and let $\sum_{k=1}^{\infty} b_k$ be a rearrangement. Let

(s_n) and (t_n) be their partial sum sequences respectively. Let $A = \lim_{n\to\infty} s_n$. We must show $t_n \to A$.

Let $\varepsilon > 0$. Since $s_n \to A$, there exists $N_1 \in \mathbb{N}$ such that $|s_n - A| < \varepsilon/2$ for all $n \geq N_1$. Since $\sum_{k=1}^{\infty} a_k$ converges absolutely, there exists $N_2 \in \mathbb{N}$ such that $|a_{m+1}| + \cdots + |a_n| < \varepsilon$ for all $n > m \geq N_2$. Let $N = \max\{N_1, N_2\}$. Choose $M \geq N$ large enough so that $\{a_1, \dots, a_N\} \subseteq \{b_1, \dots, b_M\}$. Suppose $n \geq M$. Then

$$b_1 + \cdots + b_n = a_1 + \cdots + a_N + C$$

where C is a sum of $n - N$ terms of the form a_n where $n \geq N \geq N_2$. Notice then $|C| < \varepsilon/2$ and

$$\begin{aligned} |t_n - A| &= |t_n - s_N + s_N - A| \\ &\leq |t_n - s_N| + |s_N - A| \\ &= |C| + |s_N - A| \\ &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon. \end{aligned}$$