Theorem Let $(a_n) \subseteq \mathbb{R}$ be a given sequence and let S be its set of subsequential limits.
Then $\limsup_{n\to\infty} a_n = \sup S$ and $\liminf_{n\to\infty} a_n = \inf S$.

Proof We'll do the case when (a_n) is bounded, so $\limsup a_n$ and $\liminf a_n$ are real numbers.
We'll do the proof for $\limsup a_n$.

Here's the outline of our proof:
Let $t \in S$. We'll show that

$$t \le \limsup a_n$$

This will show $\limsup a_n$ is an upper bound for S

so $\limsup a_n \ge \sup S$

Then, $\limsup a_n \le \sup S$ since $\limsup a_n \in S$ by Lemma 2.
This will show $\limsup a_n = \sup S$.

Since $t \in S$, $\exists$ (a_{n_k}) such that $\lim a_{n_k} = t$.

Let's compare $\limsup a_{n_k}$ and $\limsup a_n$.

Note $\forall$ N,

$$\sup\{a_{n_k} : k > N\} \le \sup\{a_n : n > N\}$$

proof below

So $\underbrace{\limsup_{k\to\infty} a_{n_k}}_{=t \text{ by Theorem from last time}} \le \limsup_{n\to\infty} a_n$

Observe $\{a_{n_k} : k > N\} \subseteq \{a_n : n > N\}$

since if $k > N$, then $n_k > N$, so $n_{N+1} > N$, $n_{N+2} > N, \ldots$

which means $a_{n_k} \in \{a_n : n > N\}$ for all $k > N$.

This means $\sup\{a_{n_k} : k > N\} \leq \sup\{a_n : n > N\}$

(you can show $A \subseteq B \Rightarrow \sup A \leq \sup B$)