§4.2 Functional Limits, continued

Theorem Let $f: A \to \mathbb{R}$ be a function and let $c \in \mathbb{R}$ be a limit point of A. Then $\lim_{x\to c} f(x) = L$ if and only if for every sequence $(x_n) \subseteq A \setminus \{c\}$ such that $x_n \to c$, it is the case that $f(x_n) \to L$.

Corollary 1 (Algebraic Limit Theorem for Functional Limits)

Let $f, g: A \to \mathbb{R}$ be given functions, let $c \in \mathbb{R}$ be a limit point of A and suppose $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$. Then

1. $\lim_{x\to c} kf(x) = kL$ for any $k \in \mathbb{R}$
2. $\lim_{x\to c} \big(f(x) + g(x)\big) = L + M$
3. $\lim_{x\to c} f(x)\,g(x) = LM$
4. $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$, assuming $M \neq 0$

Proof of 1 Let $k \in \mathbb{R}$. Let $(x_n) \subseteq A \setminus \{c\}$ be a sequence such that $x_n \to c$. Then $f(x_n) \to L$.

By the Algebraic Limit Theorem for sequences, $kf(x_n) \to kL$. Therefore $\lim_{x \to c} (kf)(x) = kL$.

The others are similar and left as an exercise.

Corollary 2 (Divergence Criterion) If there exist sequences $(x_n) \subseteq A \setminus \{c\}$ and $(y_n) \subseteq A \setminus \{c\}$ such that $\lim_{n \to \infty} x_n = \lim_{n \to \infty} y_n = c$ but $\lim_{n \to \infty} f(x_n) \neq \lim_{n \to \infty} f(x_n)$, then $\lim_{x \to c} f(x)$ does not exist.

Example Let
$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Prove $\lim_{x \to 0} f(x)$ does not exist.

Let $x_n = \dfrac{1}{2n\pi}$ and $y_n = \dfrac{1}{2n\pi + \frac{\pi}{2}}$.

Then $\lim_{n \to \infty} x_n = \lim_{n \to \infty} y_n = 0$. But $\lim_{n \to \infty} f(x_n) = 0$ and $\lim_{n \to \infty} f(y_n) = 1$.

The graph of this function is called the topologist's sine curve

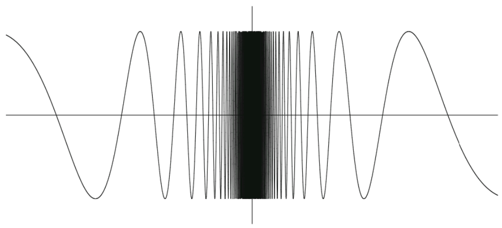

Figure 4.5: The function $\sin(1/x)$ near zero.