§ 4.3 Continuous Functions

Def A function $f : A \to \mathbb{R}$ is continuous at $c \in A$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(c)| < \varepsilon$ for all $x \in A$ such that $|x - c| < \delta$.

We say f is continuous on A if it is continuous at all $c \in A$.

Remarks ① Notice c must be a point of the domain A but it does not have to be a limit point of A.

② If c is a limit point, continuity at c is equivalent to saying $\lim_{x \to c} f(x) = f(c)$.

③ Functions are automatically continuous at their isolated points.

Example Let $f : [0, \infty) \to \mathbb{R}$ be given by $f(x) = \sqrt{x}$.

Prove f is continuous on $[0, \infty)$.

Let $c \in [0, \infty)$. Let $\varepsilon > 0$. Suppose $c = 0$ and define $\delta = \underline{\varepsilon^2}$

Let $x \in [0, \infty)$ and $|x - c| < \delta$. Then

$$|f(x) - f(c)| = \sqrt{x} < \sqrt{\delta} = \varepsilon.$$

Suppose $c > 0$ and define $\delta = \underline{\varepsilon \sqrt{c}}$.

Suppose $x \in [0, \infty)$ and $|x - c| < \delta$. Then

$$|f(x) - f(c)| = |\sqrt{x} - \sqrt{c}| \cdot \frac{\sqrt{x} + \sqrt{c}}{\sqrt{x} + \sqrt{c}}$$

$$= \frac{|x - c|}{\sqrt{x} + \sqrt{c}}$$

(show this step with two cases: ① $\sqrt{x} - \sqrt{c} \geq 0$ ② $\sqrt{x} - \sqrt{c} < 0$)

$$< \frac{\delta}{\sqrt{x} + \sqrt{c}}$$

$$\leq \frac{\delta}{\sqrt{c}}$$

$$= \varepsilon$$

Theorem (Sequential Criterion for Continuity)

Let $f : A \to \mathbb{R}$ be a given function and let $c \in A$.

Then f is continuous at c if and only if

for every sequence $(x_n) \subseteq A$ such that $x_n \to c$,

it follows that $f(x_n) \to f(c)$.

Equivalently, f is not continuous at c iff there exists

a sequence $(x_n) \subseteq A$ such that $x_n \to c$ but

$f(x_n) \not\to f(c)$.

Proof This proof is very similar to the

proof of Theorem 4.2.3. Try it as an exercise.

Theorem (Algebraic Continuity Theorem) Let $f: A \to \mathbb{R}$ and $g: A \to \mathbb{R}$ be functions that are continuous at some $c \in A$. Then

1. $kf(x)$ is continuous at c for any $k \in \mathbb{R}$.
2. $(f+g)(x) = f(x) + g(x)$ is continuous at c
3. $(fg)(x) = f(x)\, g(x)$ is continuous at c
4. $(f/g)(x) = f(x)/g(x)$ is continuous at c provided that $g(c) \neq 0$.

Proof of 1 Let $k \in \mathbb{R}$. Let $(x_n) \subseteq A$ be a sequence such that $x_n \to c$. Then $f(x_n) \to f(c)$. By the Algebraic Limit Theorem for sequences, $kf(x_n) \to kf(c)$.

Theorem (Discontinuity Criterion) Let $f: A \to \mathbb{R}$ and let $c \in A$ be given. If there exists $(x_n) \subseteq A$ such that $x_n \to c$ and $f(x_n) \not\to f(c)$, then f is not continuous at c.