§17 Continuity

Last time we saw two definitions for

$$\lim_{x \to a} f(x).$$

Today we'll use these to talk about continuity.

Def Let $f: D \to \mathbb{R}$ be a function with domain D and let $x_0 \in D$. Then f is continuous at x_0 if

$$\lim_{x \to x_0} f(x) = f(x_0).$$

Remark This means ① $\lim_{x \to x_0} f(x)$ must exist

and ② must equal $f(x_0)$.

Example Give an ε-δ proof to show that

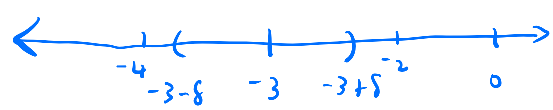
$f(x) = x^2$ is continuous at $x_0 = -3$.

Scratch work Assume $0 < |x - x_0| < \delta$ or $0 < |x+3| < \delta$

$$
\begin{aligned}
|f(x) - f(x_0)| &= |x^2 - 9| \\
&= |x-3|\,|x+3| \\
&< \delta\,|x-3|
\end{aligned}
$$

How do we bound $|x-3|$? $|x-3| \le |x| + 3$

How do we bound $|x|$? Remember $x \in (x_0 - \delta, x_0 + \delta)$



If $\delta \le 1$, then $|x| \le 4$. So $|x-3| \le 4+3 = 7$

Proof Let $\varepsilon > 0$ and define $\delta = \min\{1, \varepsilon/7\}$.

Suppose $0 < |x+3| < \delta$. Then

$$|x| = |x+3-3| \le |x+3| + 3 < \delta + 3 \le 4$$

and $\quad |f(x) - f(-3)| = |x^2 - 9|$

$$
\begin{aligned}
&= |x-3||x+3| \\
&< \delta |x-3| \\
&\le \delta \left(|x| + 3 \right) \\
&< 7\delta \\
&\le \varepsilon
\end{aligned}
$$