§ 4.4 Continued

Def A function $f: A \to \mathbb{R}$ is <u>uniformly continuous on A</u> if for each $\varepsilon > 0$ there exists $\delta > 0$ such that when $x, y \in A$ with $|x-y| < \delta$ it follows that $|f(x) - f(y)| < \varepsilon$.

Examples

① $f(x) = 3x+1$ is uniformly continuous on $\mathbb{R}$.

② $f(x) = x^2$ is uniformly continuous on $[-4, 3]$

③ $f(x) = \frac{1}{x}$ is uniformly continuous on $(2, 3)$

④ $f(x) = \frac{1}{x-3}$ is uniformly continuous on $(6, \infty)$

② Let $\varepsilon > 0$. Define $\delta = \underline{\varepsilon/8}$. Suppose $x, y \in [-4, 3]$ and $|x-y| < \delta$. Then

$$
\begin{aligned}
|f(x) - f(y)| &= |x^2 - y^2| \\
&= |x-y||x+y| \\
&< \delta |x+y|
\end{aligned}
$$

$$\leq \delta(|x| + |y|)$$

$$\leq 8\delta$$

$$= \varepsilon.$$

③ Let $\varepsilon > 0$. Define $\delta = \underline{4\varepsilon}$. Suppose $x, y \in (2,3)$ and $|x-y| < \delta$. Then

$$|f(x) - f(y)| = \left| \frac{1}{x} - \frac{1}{y} \right|$$

$$= \left| \frac{y-x}{xy} \right|$$

$$< \frac{\delta}{|x||y|}$$

$$< \frac{\delta}{4}$$

$$= \varepsilon.$$

④ Let $\varepsilon > 0$. Define $\delta = \underline{9\varepsilon}$. Suppose $x, y \in (6, \infty)$ and $|x-y| < \delta$. Then

$$|f(x) - f(y)| = \left| \frac{1}{x-3} - \frac{1}{y-3} \right|$$

$$= \left| \frac{(y-3)-(x-3)}{(x-3)(y-3)} \right|$$

$$< \frac{\delta}{|x-3||y-3|}$$

$$< \frac{\delta}{9}$$

$$= \varepsilon.$$

Theorem (Sequential Criterion for Absence of Uniform Continuity) A function $f: A \to \mathbb{R}$ fails to be uniformly continuous on A if and only if there exists $\varepsilon_0 > 0$ and sequences $(x_n), (y_n) \subseteq A$ such that $|x_n - y_n| \to 0$ and $|f(x_n) - f(y_n)| \geq \varepsilon_0$.

Proof to come.

Examples ① $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$.

Let $x_n = n$ and $y_n = n + \frac{1}{n}$ and $\varepsilon_0 = \underline{\ 2\ }$.

Then $|x_n - y_n| \to 0$ and for all $n \in \mathbb{N}$,

$$
\begin{aligned}
|f(x_n) - f(y_n)| &= \left| n^2 - \left(n + \tfrac{1}{n}\right)^2 \right| \\
&= \left| n^2 - \left(n^2 + 2 + \tfrac{1}{n^2}\right) \right| \\
&= 2 + \frac{1}{n^2} \\
&\geq 2 \\
&= \varepsilon_0 .
\end{aligned}
$$

② $f(x) = \sin\left(\frac{1}{x}\right)$ is not uniformly continuous on $(0, \infty)$.

Let $x_n = \dfrac{1}{\frac{\pi}{2} + 2n\pi}$, $y_n = \dfrac{1}{2n\pi}$, and $\varepsilon_0 = \underline{\ 1\ }$.

Then $|x_n - y_n| \to 0$ and for all $n \in \mathbb{N}$,

$$|f(x_n) - f(y_n)| = 1 = \varepsilon_0$$