§4.4 Uniform continuity wrap-up.

Theorem (Sequential Criterion for Absence of Uniform Continuity) A function $f: A \to \mathbb{R}$ fails to be uniformly continuous on A if and only if there exists $\varepsilon_0 > 0$ and sequences $(x_n), (y_n) \subseteq A$ such that
$|x_n - y_n| \to 0$ and $|f(x_n) - f(y_n)| \geq \varepsilon_0$.

Proof ($\Rightarrow$) Suppose f is not uniformly continuous on A. Then there exists $\varepsilon_0 > 0$ such that for each $\delta > 0$ there exist $x, y \in A$ such that $|x-y| < \delta$ and $|f(x) - f(y)| \geq \varepsilon_0$. Therefore, for each $n \in \mathbb{N}$, there exist $x_n, y_n \in A$ such that $|x_n - y_n| < \frac{1}{n}$ and $|f(x_n) - f(y_n)| \geq \varepsilon_0$. It follows that we have sequences (x_n) and (y_n) with the desired properties.

($\Leftarrow$) Exercise.

Theorem If $f: A \to \mathbb{R}$ is uniformly continuous on A then it is continuous on A.

The converse is not true in general, but we'll see that it is true when K is compact.

Theorem If $f: K \to \mathbb{R}$ is continuous on K and K is compact, then f is uniformly continuous on K.

Proof Suppose f is not uniformly continuous on K. Then there exists $\varepsilon_0 > 0$ and sequences $(x_n), (y_n) \subseteq K$ such that $|x_n - y_n| \to 0$ and $|f(x_n) - f(y_n)| \geq \varepsilon_0$ for all $n \in \mathbb{N}$. Since K is compact, there exist convergent subsequences (x_{n_k}) and (y_{n_k}) of (x_n) and (y_n) with limits $x, y \in K$ respectively. Notice that $|x - y| = \lim_{k \to \infty} |x_{n_k} - y_{n_k}| = 0$, which implies $x = y$.

Since f is continuous,

$$|f(x_{n_k}) - f(y_{n_k})| \to |f(x) - f(y)| = 0$$

Therefore there exists $N \in \mathbb{N}$ such that

$$\varepsilon_0 \leq |f(x_{n_k}) - f(y_{n_k})| < \varepsilon_0$$

for all $k \geq N$, a contradiction.

Theorem If $f : A \to \mathbb{R}$ is uniformly continuous on A and A is bounded, then $f(A)$ is bounded.

Proof Exercise.

Theorem If $f : A \to \mathbb{R}$ is uniformly continuous on A and $(x_n) \subseteq A$ is a Cauchy sequence, then $(f(x_n))$ is a Cauchy sequence.

Proof Exercise.

Example Give 3 different proofs that $f(x) = \frac{1}{x}$ is not uniformly continuous on $(0,1)$.