§19 Uniform continuity

Theorem If f is continuous on $[a,b]$ then f is uniformly continuous on $[a,b]$.

Proof Assume f is not uniformly continuous on $[a,b]$. Then $\exists\ \varepsilon>0,\ \forall\ \delta>0,\ \exists\ x,y\in[a,b]$ such that $|x-y|<\delta$ but $|f(x)-f(y)|\geq\varepsilon$.

Therefore $\forall\ n\geq 1,\ \exists\ x_n, y_n\in[a,b]$ such that $|x_n-y_n|<\frac{1}{n}$ but $|f(x_n)-f(y_n)|\geq\varepsilon$.

Notice that $\lim|x_n-y_n|=0$. Moreover, $\exists$ subsequences (x_{n_k}) and (y_{n_k}) that converge to $x_0, y_0\in[a,b]$. In fact,

$$0 = \lim |x_{n_k} - y_{n_k}| = |x_0 - y_0|$$

so $x_0 = y_0$. Therefore $\lim x_{n_k} = \lim y_{n_k} = x_0$

so $\lim f(x_{n_k}) = \lim f(y_{n_k})$ since

f is continuous. However, this implies

$$\varepsilon \leq \lim \left| f(x_{n_k}) - f(y_{n_k}) \right| = 0$$

which is a contradiction since $\varepsilon > 0$.

<u>Theorem</u> If $\exists\ \varepsilon > 0$ and sequences $(x_n), (y_n) \subseteq D$ such that $\lim |x_n - y_n| = 0$ and $|f(x_n) - f(y_n)| \geq \varepsilon$ for all $n \geq 1$, then f is not uniformly continuous on D.

<u>Proof</u> We must show $\exists\ \varepsilon > 0$ such that $\forall\ \delta > 0\ \exists\ x, y \in D$ such that $|x - y| < \delta$ but $|f(x) - f(y)| \geq \varepsilon$.

Let ε be the value given in our hypothesis and (x_n) and (y_n) the given sequences.

Let $\delta > 0$. We must show $\exists\ x, y \in D$ such that $|x - y| < \delta$ but $|f(x) - f(y)| \geq \varepsilon$.

Since $\lim |x_n - y_n| = 0$, $\exists\ N$ such that

$$|x_n - y_n| < \delta$$

for all $n > N$. Moreover

$$|f(x_n) - f(y_n)| \geq \varepsilon \quad \text{for all } n.$$

Therefore any elements of our sequences like x_{N+1} or y_{N+1} (or greater index values) are the desired $x, y \in D$.