§ 1.3 The Axiom of Completeness

Definition A nonempty set $A \subseteq \mathbb{R}$ is said to be

1. bounded above if there exists $M \in \mathbb{R}$ such that $x \le M$ for all $x \in A$. We call M an upper bound of A.

2. bounded below if there exists $m \in \mathbb{R}$ such that $x \ge m$ for all $x \in A$. We call m a lower bound of A.

3. bounded if it is bounded above and bounded below.

Examples Decide whether each of the following is bounded above, bounded below, or bounded. For those bounded above give 2 examples of upper bounds. Similarly give 2 examples of lower bounds for those that are bounded above.

1. $A = \{x \in \mathbb{R} : x^2 \le 5\} = [-\sqrt{5}, \sqrt{5}]$
2. $B = \{x \in \mathbb{R} : x^3 < 5\} = (-\infty, 5^{1/3})$
3. $C = \{x \in \mathbb{N} : x \le 5\} = \{1, 2, 3, 4, 5\}$
4. $D = \{x \in \mathbb{Q} : x^2 \le 2\} = [-\sqrt{2}, \sqrt{2}] \cap \mathbb{Q}$
5. $E = \{x \in \mathbb{Q} : x^2 < 2\} = (-\sqrt{2}, \sqrt{2}) \cap \mathbb{Q}$

<u>Definition</u> A real number M is a <u>maximum</u> of a set $A \subseteq \mathbb{R}$ if $x \le M$ for all $x \in A$ <u>and</u> $M \in A$. A real number m is a <u>minimum</u> of A if $x \ge m$ for all $x \in A$ <u>and</u> $m \in A$.

<u>Example</u> Which sets in the previous example have a maximum? A minimum? If the max/min exists, what is it?

① $\max A = \sqrt{5}$, $\min A = -\sqrt{5}$

③ $\max C = 5$, $\min C = 1$

②, ④, ⑤ no max, no min

<u>Definition</u> Let $A \subseteq \mathbb{R}$ be a nonempty set that is bounded above. Then $U \in \mathbb{R}$ is said to be the <u>supremum</u> or <u>least upper bound</u> of A if

① $x \le U$ for all $x \in A$

② If M is an upper bound of A, then $U \le M$.

We denote the supremum of A by $\sup A$. If A is not bounded above, we let $\sup A = +\infty$.

<u>Examples</u> Which sets in the previous examples have a supremum? If it exists, what is it?

① $\sup A = \sqrt{5}$, ② $\sup B = 5^{1/3}$, ③ $\sup C = 5$, ④, ⑤ $\sup D = \sup E = \sqrt{2}$

Distinguishing $\mathbb{Q}$ from $\mathbb{R}$.

Notice the set $\{ r \in \mathbb{Q} : r^2 < 2 \}$ is non-empty and bounded above and has a supremum ($\sqrt{2}$).

However its supremum is not a rational number; we need more than just rationals to define the supremum of the set despite the fact that it only contains rationals. In this sense $\mathbb{Q}$ is "incomplete."

Axiom of Completeness Every nonempty set of real numbers that is bounded above has a least upper bound.

"Informal" Definition of $\mathbb{R}$ The real numbers consist of $\mathbb{Q}$ along with suprema of nonempty sets of rationals that are bounded above.

"Formal" Definition of $\mathbb{R}$ It is an ordered field (number system with addition, mult., inverses, $<$) that contains $\mathbb{Q}$ and satisfies the Axiom of Completeness.

Example Let $A \subseteq \mathbb{R}$ be a non-empty set that is bounded above and let $c \in \mathbb{R}$. Define the set B by

$$B = \{c + a : a \in A\}.$$

Prove that $\sup B = c + \sup A$.

We will first show that $c + \sup A$ is an upper bound of B. Let $b \in B$. Then $b = c + a$ for some $a \in A$. Since $a \leq \sup A$, $b = c + a \leq c + \sup A$. Since b was arbitrary, $c + \sup A$ is an upper bound of B.

Next, we will show $c + \sup A$ is the least upper bound of B. Let U be an arbitrary upper bound of B. We must show $c + \sup A \leq U$. To show this, we will show $\sup A \leq U - c$ by showing $U - c$ is an upper bound of A. Let $a \in A$. Then $c + a \in B$ and so $c + a \leq U$. Therefore $a \leq U - c$, which shows $U - c$ is an upper bound of A. Since $\sup A$ is the least upper bound of A, $\sup A \leq U - c$.

Lemma Let $S \subseteq \mathbb{R}$ be a set that is bounded above and let $\alpha \in \mathbb{R}$ be an upper bound for S. Then $\alpha = \sup S$ if and only if for every $\varepsilon > 0$, there exists $x \in S$ such that $x > \alpha - \varepsilon$.

Proof ($\Rightarrow$) Assume $\alpha = \sup S$. Let $\varepsilon > 0$. Then $\alpha - \varepsilon < \alpha$, so $\alpha - \varepsilon$ is not an upper bound for S (since α is the least upper bound). Therefore there exists $x \in S$ such that $x > \alpha - \varepsilon$.

($\Leftarrow$) Assume $\forall\, \varepsilon > 0\ \exists\, x \in S$ such that $x > \alpha - \varepsilon$. Let u be an upper bound for S. We must prove $\alpha \leq u$. Suppose $u < \alpha$ and let $\varepsilon_0 = \alpha - u$. Then there exists $x \in S$ such that

$$x > \alpha - \varepsilon_0 = \alpha - (\alpha - u) = u$$

but this contradicts that u is an upper bound.

Problem 1. Let S and T be non-empty, bounded subsets of $\mathbb{R}$ and suppose that $S \subseteq T$. Prove that $\sup S \leq \sup T$.

We'll prove that $\sup T$ is an upper bound of S. This will imply $\sup S \leq \sup T$ since $\sup S$ is the least upper bound of S.

Let $x \in S$. Then $x \in T$. This implies $x \leq \sup T$ since $\sup T$ is an upper bound of T. Therefore $\sup T$ is an upper bound of S.

Problem 2. Let $A, B \subseteq \mathbb{R}$ be given sets that are nonempty and bounded above. Define $C = \{a + b : a \in A, b \in B\}$. Use today's lemma to prove that $\sup C = \sup A + \sup B$.

Let $\varepsilon > 0$. We must show there exists $c \in C$ such that $c > \sup A + \sup B - \varepsilon$. Observe that by the Lemma, there exists $a \in A$ such that $a > \sup A - \frac{\varepsilon}{2}$. Similarly, there exists $b \in B$ such that $b > \sup B - \frac{\varepsilon}{2}$. Let $c = a + b$. Then $c \in C$ and

$$c = a + b > \sup A - \varepsilon/2 + \sup B - \varepsilon/2$$
$$= \sup A + \sup B - \varepsilon.$$

Problem 3. For each set below, give its maximum and minimum (if they exist), as well as its supremum and infimum.

a. The half open interval $(-2, 4] = \{x \in \mathbb{R} : -2 < x \leq 4\}$.

b. $\{1/n : n \in \mathbb{N}\}$

c. $\bigcap_{n=1}^{\infty}(-1/n, 1 + 1/n)$

d. $\left\{r \in \mathbb{Q} : 0 < r^2 \leq 2\right\}$

e. $\left\{r \in \mathbb{R} : 0 < r^2 \leq 2\right\}$

(a) min DNE, inf $= -2$, max $=$ sup $= 4$

(b) min DNE, inf $= 0$, max $=$ sup $= 1$

(c) set is $[0, 1]$, so min $=$ inf $= 0$, max $=$ sup $= 1$

(d) min DNE, inf $= 0$, max DNE, sup $= \sqrt{2}$

(e) min DNE, inf $= 0$, max $=$ sup $= \sqrt{2}$