Theorem Let (f_n) be a sequence of functions. Then $f_n \to f$ uniformly on A if and only if

$$\lim_{n\to\infty} \sup\{|f_n(x)-f(x)| : x\in A\}=0.$$

Proof $(\Rightarrow)$ Let $(\alpha_n)\subseteq\mathbb{R}$ be given by

$$\alpha_n = \sup\{|f_n(x)-f(x)| : x\in A\}$$

for each $n\in\mathbb{N}$. We must show $\alpha_n \to 0$. Let $\varepsilon>0$. Since $f_n\to f$ uniformly on A, there exists $N\in\mathbb{N}$ such that $|f_n(x)-f(x)|<\frac{\varepsilon}{2}$ for all $n\geq N$ and all $x\in A$.

Suppose $n\geq N$. Since $|f_n(x)-f(x)|<\frac{\varepsilon}{2}$ for all $x\in A$, $\frac{\varepsilon}{2}$ is an upper bound for the set $\{|f_n(x)-f(x)| : x\in A\}$. Therefore $\alpha_n\leq\frac{\varepsilon}{2}<\varepsilon$.

$(\Leftarrow)$ Exercise.

Example Let $f_n(x) = \dfrac{x^2 + nx}{n}$ and $f(x) = x$.

Prove that (f_n) does not converge uniformly to f on $\mathbb{R}$.

Observe that $|f_n(x) - f(x)| = \dfrac{x^2}{n}$ for all $x \in \mathbb{R}$, $n \in \mathbb{N}$.

Therefore $\sup\left\{ \dfrac{x^2}{n} : x \in \mathbb{R} \right\} = \infty$ for all $n \in \mathbb{N}$.

This means $\lim\limits_{n \to \infty} \sup\{ |f_n(x) - f(x)| : x \in \mathbb{R} \} \neq 0$.

Theorem (Cauchy criterion for uniform converges)

Let (f_n) be a sequence of functions. Then (f_n) converges uniformly (to some f) on A if and only if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$|f_m(x) - f_n(x)| < \varepsilon$ for all $n, m \geq N$ and all $x \in A$.

Proof. $(\Longrightarrow)$ Let $\varepsilon > 0$. Since $f_n \to f$ uniformly on A, there exists $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \varepsilon/2$ for all $n \geq N$ and $x \in A$. Let $n, m \geq N$ and $x \in A$. Then

$$
\begin{aligned}
|f_n(x) - f_m(x)| &= |f_n(x) - f(x) + f(x) - f_m(x)| \\
&\leq |f_n(x) - f(x)| + |f_m(x) - f(x)| \\
&< \varepsilon/2 + \varepsilon/2 \\
&= \varepsilon.
\end{aligned}
$$

($\Leftarrow$) It is straightforward to see that under the given assumption, $(f_n(x)) \subseteq \mathbb{R}$ is a Cauchy sequence for each $x \in A$. Therefore $(f_n(x))$ converges for each $x \in \mathbb{R}$, and we can define $f: A \to \mathbb{R}$ by

$$f(x) = \lim_{n\to\infty} f_n(x).$$

We must prove $f_n \to f$ uniformly. Let $\varepsilon > 0$. Then there exists $N \in \mathbb{N}$ such that $|f_n(x) - f_m(x)| < \varepsilon/2$ for all $m, n \geq N$ and all $x \in A$.

Let $n \geq N$ and $x \in A$. Since $|f_n(x) - f_m(x)| < \varepsilon/2$ for all $m \geq N$, by the Order Limit Theorem,

$$\lim_{m\to\infty} |f_n(x) - f_m(x)| \leq \frac{\varepsilon}{2} < \varepsilon.$$

Since $\lim_{m\to\infty} |f_n(x) - f_m(x)| = |f_n(x) - f(x)|$, we have that $|f_n(x) - f(x)| < \varepsilon$.