§ 6.3 Uniform convergence and differentiation

Example Consider $f_n(x) = x^{1+\frac{1}{2n-1}}$. Notice that for each $x \in [-1, 1]$,

$$f(x) = \lim_{n\to\infty} f_n(x) = x \lim_{n\to\infty} x^{\frac{1}{2n-1}} = |x|$$

since $x^{\frac{1}{2n-1}} \to -1$ as $n \to \infty$ when $x < 0$ and $x^{\frac{1}{2n-1}} \to 1$ as $n\to\infty$ when $x > 0$. So f_n is differentiable at 0 for each $n \in \mathbb{N}$ but f is not differentiable at 0.

Theorem (Differentiable Limit Theorem) Let $f_n \to f$ pointwise on $[a,b]$ and assume each f_n is differentiable on $[a,b]$. Suppose $f_n' \to g$ uniformly on $[a,b]$ for some function g. Then f is differentiable on $[a,b]$ and $f' = g$.

Fake proof

$$\lim_{x\to c} \frac{f(x)-f(c)}{x-c} = \lim_{x\to c} \lim_{n\to\infty} \frac{f_n(x)-f_n(c)}{x-c}$$

$$\stackrel{?}{=} \lim_{n\to\infty} \lim_{x\to c} \frac{f_n(x)-f_n(c)}{x-c} = \lim_{n\to\infty} f_n'(c) = g(c)$$

<u>Proof</u> We must show that for each $c \in [a,b]$

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = g(c).$$

Let $c \in [a,b]$. We must show for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\left| \frac{f(x) - f(c)}{x - c} - g(c) \right| < \varepsilon$$

whenever $x \in [a,b]$ and $0 < |x - c| < \delta$. Let $\varepsilon > 0$.

Observe that for any $x \neq c$ and $n \in \mathbb{N}$,

$$\left| \frac{f(x) - f(c)}{x - c} - g(c) \right|$$

$$= \left| \frac{f(x) - f(c)}{x - c} - \frac{f_n(x) - f_n(c)}{x - c} + \frac{f_n(x) - f_n(c)}{x - c} - f_n'(c) + f_n'(c) - g(c) \right|$$

$$\leq \left| \frac{f(x) - f(c)}{x - c} - \frac{f_n(x) - f_n(c)}{x - c} \right| + \left| \frac{f_n(x) - f_n(c)}{x - c} - f_n'(c) \right| + \left| f_n'(c) - g(c) \right|. \quad (1)$$

We will choose a particular value of n now.

Since $f_n' \to g$ uniformly, there exists $N_1 \in \mathbb{N}$ such that

$$|f_n'(c) - g(c)| < \varepsilon/3 \quad (2)$$

for all $n \geq N_1$.

Since f_n' converges uniformly, there exists $N_2 \in \mathbb{N}$ so that $|f_m'(x) - f_n'(x)| < \varepsilon/3$ for all $m, n \geq N_2$ and all $x \in [a,b]$. Let $N = \max\{N_1, N_2\}$. Since f_N is differentiable at c, there exists $\delta > 0$ such that

$$\left| \frac{f_N(x) - f_N(c)}{x - c} - f_N'(c) \right| < \varepsilon/3 \quad (3)$$

for all $x \in [a,b]$ such that $0 < |x - c| < \delta$.

Suppose $m \geq N$ and $x \in [a,b]$ is such that $0 < |x-c| < \delta$. By the Mean Value Theorem applied to $f_m - f_N$, there exists α between x and c such that

$$f_m'(\alpha) - f_N'(\alpha) = \frac{(f_m(x) - f_N(x)) - (f_m(c) - f_N(c))}{x - c}$$

$$= \frac{f_m(x) - f_m(c)}{x - c} - \frac{f_N(x) - f_N(c)}{x - c}.$$

Therefore

$$\left| \frac{f_m(x) - f_m(c)}{x - c} - \frac{f_N(x) - f_N(c)}{x - c} \right| = \left| f_m'(\alpha) - f_N'(\alpha) \right|$$

$$< \varepsilon/3 .$$

By the Order Limit Theorem

$$\lim_{m \to \infty} \left| \frac{f_m(x) - f_m(c)}{x - c} - \frac{f_N(x) - f_N(c)}{x - c} \right| \leq \varepsilon/3 .$$

Since $\lim_{m \to \infty} f_m(x) = f(x)$ for all $x \in [a, b]$, we have

$$\left| \frac{f(x) - f(c)}{x - c} - \frac{f_N(x) - f_N(c)}{x - c} \right| \leq \varepsilon/3 . \quad (4)$$

Therefore, plugging (2), (3), (4) into (1) with $n = N$, the result follows.