§6.4 Series of Functions

Def Let (f_n) and f be functions defined on $A \subseteq \mathbb{R}$. For each $n \in \mathbb{N}$, let $s_n = f_1 + \cdots + f_n$. The series $\sum_{n=1}^{\infty} f_n$ converges pointwise to f on A if $s_n \to f$ pointwise on A. Likewise, $\sum_{n=1}^{\infty} f_n$ converges uniformly to f on A if $s_n \to f$ uniformly on A. We often write $f(x) = \sum_{n=1}^{\infty} f_n(x)$.

Theorem (Term-by-term continuity) Suppose (f_n) is a sequence of continuous functions on A such that $\sum_{n=1}^{\infty} f_n$ converges uniformly to some function f on A. Then f is continuous.

Theorem (Term-by-term differentiability) Suppose (f_n) is a sequence of functions differentiable on an interval A such that $\sum_{n=1}^{\infty} f_n'$ converges uniformly on A and there exists $x_0 \in A$ such that $\sum_{n=1}^{\infty} f_n(x_0)$ converges. Then $\sum_{n=1}^{\infty} f_n$ converges to some differentiable function f on A and

$$f' = \sum_{n=1}^{\infty} f_n'$$

Theorem (Cauchy criterion) A series $\sum_{n=1}^{\infty} f_n$ converges uniformly on A if and only if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$|f_{m+1}(x) + \cdots + f_n(x)| < \varepsilon$$

for all $n > m \geq N$ and $x \in A$.

Theorem (Weierstrass M-test) Let (f_n) be a sequence of functions defined on A. Suppose for each $n \in \mathbb{N}$ there exists $M_n > 0$ such that $|f_n(x)| \leq M_n$ for all $x \in A$. If $\sum_{n=1}^{\infty} M_n$ converges, then $\sum_{n=1}^{\infty} f_n$ converges uniformly on A.

Proof By the Cauchy criterion for uniform convergence, we must show that for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$|f_{m+1}(x) + \cdots + f_n(x)| < \varepsilon$$

for all $n > m \geq N$ and $x \in A$. Let $\varepsilon > 0$. Since $\sum_{n=1}^{\infty} M_n$ converges, there exists $N \in \mathbb{N}$ such that

$$M_{m+1} + \cdots + M_n < \varepsilon$$

for all $n > m \geq N$. Let $n > m \geq N$ and $x \in A$. Then

$$\begin{aligned} |f_{m+1}(x) + \cdots + f_n(x)| &\leq |f_{m+1}(x)| + \cdots + |f_n(x)| \\ &\leq M_{m+1} + \cdots + M_n \\ &< \varepsilon. \end{aligned}$$