# Math 301 — Exam 3 review guide solutions

**Solution 1.**

a. *True*: the function $g(x) = x^{1/3}$ is continuous at 0, which implies that $f = g \circ f^3$ is continuous at 0.

b. *False*: consider the function $f$ where $f(x) = 1$ when $x \geq 0$ and $f(x) = 0$ when $x < 0$, which is not continuous on $\mathbb{R}$. For every compact set $K$, we have that $f(K)$ is either $\{0\}, \{1\}$ or $\{0, 1\}$ all of which are compact. *Modified statement that is true:* the converse.

c. *False*: consider $f(x) = |x|$ which is continuous but not differentiable at 0. *Modified statement that is true:* the converse.

d. *True*: this is part of the statement of the Algebraic Differentiability Theorem.

e. *False*: consider the functions $f_n(x) = x^n$ for each $n \in \mathbb{N}$ defined on the set $A = [0, 1]$. Since $(f_n(x)) \subseteq \mathbb{R}$ is a Cauchy sequence for each $x \in [0, 1]$ it follows that for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|f_n(x) - f_m(x)| < \epsilon$ for all $n, m \geq N$. However, $(f_n)$ does not converge uniformly on $[0, 1]$ since each $f_n$ is continuous $[0, 1]$ but the limit function $f$ (piecewise defined as $f(x) = 0$ for all $x \in [0, 1)$ and $f(1) = 1$) is not continuous on $[0, 1]$. *Modified statement that is true:* If for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|f_n(x) - f_m(x)| < \epsilon$ for all $n, m \geq N$ and $x \in A$ then $(f_n)$ converges uniformly on $A$.

**Solution 2.**

a. Let $(x_n) \subseteq V$ be a convergent sequence with $x_n \to a$. We must show $a \in V$ which means showing that $f(a) = 0$. Since $f$ is continuous on $\mathbb{R}$ we have that $f(x_n) \to f(a)$. Since $x_n \in V$ we have that $f(x_n) = 0$ for all $n \in \mathbb{N}$ and it follows that $f(x_n) \to 0$. Therefore $f(a) = 0$ by the uniqueness of limits.

b. We first note that since $f$ is continuous on $[0, 1]$ and continuous functions on compact sets are uniformly continuous, we have that $f$ is uniformly continuous on $[0, 1]$. Let $\epsilon > 0$. Since $f$ is uniformly continuous on $[0, 1]$ there exists $\delta_1 > 0$ such that $|f(x) - f(y)| < \epsilon/2$ for all $x, y \in [0, 1]$ such that $|x - y| < \delta_1$. Since $f$ is uniformly continuous on $[1, \infty)$ there exists $\delta_2 > 0$ such that $|f(x) - f(y)| < \epsilon/2$ for all $x, y \in [1, \infty)$ such that $|x - y| < \delta_2$. Let $\delta = \min\{\delta_1, \delta_2\}$ and suppose that $x, y \in [0, \infty)$ are such that $|x - y| < \delta$. If $x, y$ are both in $[0, 1]$ or are both in $[1, \infty)$ then it is clear that $|f(x) - f(y)| < \epsilon$. Suppose that $x \in [0, 1]$ and $y \in [1, \infty)$. Then it is straightforward to see that $|x - 1| < \delta \leq \delta_1$ and $|1 - y| < \delta \leq \delta_2$. Therefore

$$\begin{aligned}
|f(x) - f(y)| &= |f(x) - f(1) + f(1) - f(y)| \\
&\leq |f(x) - f(1)| + |f(1) - f(y)| \\
&< \epsilon/2 + \epsilon/2 \\
&= \epsilon.
\end{aligned}$$

c. We claim that $\lim_{x\to 0} f(x)/x = 0$. Let $\epsilon > 0$. Define $\delta = \min\left\{\beta, \epsilon^{1/(\alpha-1)}\right\}$. Suppose that $x \in \mathbb{R}$ and $0 < |x| < \delta$. Observe that

$$\left|\frac{f(x)}{x}\right| \leq \frac{|x|^\alpha}{|x|} = |x|^{\alpha-1} < \delta^{\alpha-1} \leq \epsilon.$$

d. Since $f$ is differentiable at 0, it is continuous at 0. Therefore $f(x_n) \to f(0)$. Since $f(x_n) = 0$ for all $n \in \mathbb{N}$ it follows that $f(x_n) \to 0$. Therefore $f(0) = 0$ by the uniqueness of limits. Next we show that $f'(0) = 0$. Since $f$ is differentiable at 0 we know that the limit $\lim_{x\to 0} f(x)/x$ exists and equals $f'(0)$. Therefore, for any sequence $(a_n) \subseteq A$ such that $a_n \to 0$ with $a_n \neq 0$ for all $n \in \mathbb{N}$ we have that $f'(0) = \lim_{n\to\infty} f(a_n)/a_n$. This means that

$$f'(0) = \lim_{n\to\infty} \frac{f(x_n)}{x_n} = 0$$

since $f(x_n) = 0$ for all $n \in \mathbb{N}$. Finally we show that $f''(0) = 0$. Note that by the Mean Value Theorem, for each $n \in \mathbb{N}$ there exists $c_n \in A$ between 0 and $x_n$ such that $f'(c_n) = f(x_n)/x_n$. Since $x_n \to 0$ it follows that $c_n \to 0$. Indeed, if $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_n| < \epsilon$ for all $n \geq N$. Therefore, when $n \geq N$ we have that $|c_n| < |x_n| < \epsilon$. Since $f''(0)$ exists and $f'(0) = 0$ we have that

$$f''(0) = \lim_{n\to\infty} \frac{f'(c_n)}{c_n} = \lim_{n\to\infty} \frac{f(x_n)}{x_n c_n} = 0$$

since $f(x_n) = 0$ for all $n \in \mathbb{N}$.

e. Let $g(x) = x/2$. Let $M > 0$. We claim that $g_n \to g$ uniformly on $[-M, M]$. Let $\epsilon > 0$. Define $N \in \mathbb{N}$ to be such that $N > M^2/(2\epsilon)$. Suppose $x \in [-M, M]$ and $n \geq N$. Observe that

$$\begin{aligned}
|g_n(x) - g(x)| &= \left| \frac{nx + x^2}{2n} - \frac{x}{2} \right| \\
&= \frac{x^2}{2n} \\
&\leq \frac{M^2}{2n} \\
&\leq \frac{M^2}{2N} \\
&< \epsilon.
\end{aligned}$$

To see that $g_n \not\to g$ uniformly on $\mathbb{R}$ observe that $\sup\{|g_n(x) - g(x)| : x \in \mathbb{R}\} = \infty$. Therefore, since $\lim_{n\to\infty} \sup\{|g_n(x) - g(x)| : x \in \mathbb{R}\} \neq 0$ the claim follows.