## §1.5 Cardinality

**Def** A function $f: A \to B$ is one-to-one (or 1-1) if $a_1, a_2 \in A$, $a_1 \neq a_2$ implies $f(a_1) \neq f(a_2)$. It is called onto if for every $b \in B$, there exists $a \in A$ such that $f(a) = b$. If $f$ is both 1-1 and onto it is called a bijection.

The cardinality of a set is a term we use for the number of elements of the set. For finite sets this is a natural number and involves counting. For infinite sets there are some further subtleties.

**Def** Two sets $A$ and $B$ have the same cardinality if there exists a bijection $f: A \to B$. When this is the case we write $A \sim B$.

Examples

① $\mathbb{N} \sim E = \{2n : n \in \mathbb{N}\}$.

(it can be shown $f : \mathbb{N} \to E$ given by $f(n) = 2n$ is a bijection)

② $\mathbb{N} \sim \mathbb{Z}$

(it can be shown $f : \mathbb{N} \to \mathbb{Z}$ given by

$$f(n) = \begin{cases} \frac{n-1}{2} & \text{if } n \text{ is odd} \\ -\frac{n}{2} & \text{if } n \text{ is even} \end{cases}$$

is a bijection)

③ $(-\frac{\pi}{2}, \frac{\pi}{2}) \sim \mathbb{R}$ ($f(x) = \tan(x)$ is a bijection)

**Def** An infinite set $A$ is countable if $\mathbb{N} \sim A$.

Otherwise it is called uncountable.

**Theorem** (i) $\mathbb{Q}$ is countable (ii) $\mathbb{R}$ is uncountable.

**Proof of (ii)** Suppose $\mathbb{R}$ is countable. Then there exists a bijection $f : \mathbb{N} \to \mathbb{R}$ and we have that

(i) $x_1 = f(1), x_2 = f(2), \ldots$ is an infinite list of real numbers (unique due to 1-1)

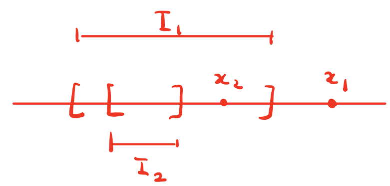
(ii) $\mathbb{R} = \{x_1, x_2, \ldots\}$ since $f$ is onto.

Let $I_1$ be a closed interval such that $x_1 \notin I_1$

Let $I_2$ be a closed interval such that

(i) $I_1 \supseteq I_2$

(ii) $x_2 \notin I_2$




Continue in this way so that

(i) $I_n \supseteq I_{n+1}$

(ii) $x_{n+1} \notin I_{n+1}$.

Then we have collection of nested closed intervals and there exists $x \in \mathbb{R}$ such that $x \in \bigcap_{n=1}^{\infty} I_n$ by the Nested Interval Property. However, $x = x_{n_0}$ for some $n_0 \in \mathbb{N}$ and so $x \notin I_{n_0}$ which contradicts that $x \in \bigcap_{n=1}^{\infty} I_n$.