§2.2 The Limit of a Sequence

Def A sequence is a function $f : \mathbb{N} \to \mathbb{R}$.

Notation: given $f : \mathbb{N} \to \mathbb{R}$, we use $x_n = f(n)$ to denote the nth term of the sequence. We denote the entire sequence by (x_n) or $(x_n)_{n=1}^{\infty}$. Parentheses are used (as opposed to braces $\{ \}$) to indicate this is an ordered collection of numbers.

Examples

1. $\left(\frac{1}{n}\right)_{n=1}^{\infty} = \left(1, \frac{1}{2}, \frac{1}{3}, \dots\right)$
2. (a_n) where $n = 2^n$ for each $n \in \mathbb{N}$
3. (x_n) where $x_1 = 2$ and $x_{n+1} = \frac{x_n + 1}{2}$.

Def We say a sequence (x_n) converges to $L \in \mathbb{R}$ if for each $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

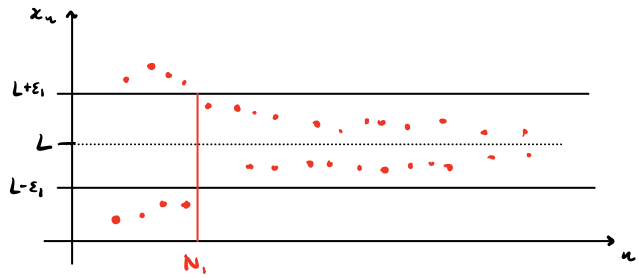
$$|x_n - L| < \varepsilon \text{ for all } n \geq N.$$

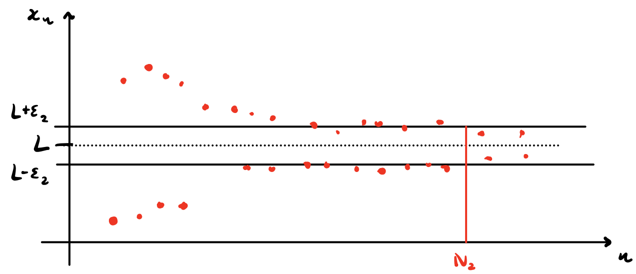
We call L the limit of the sequence.

We write $x_n \to L$ as $n \to \infty$ or $\lim_{n \to \infty} x_n = L$.

We say (x_n) converges if $x_n \to L$ for some $L \in \mathbb{R}$.

If no such L exists, we say (x_n) diverges.





Def Given $L \in \mathbb{R}$ and $\varepsilon > 0$, the set

$$V_\varepsilon(L) = \{ x \in \mathbb{R} : |x - L| < \varepsilon \}$$

is called the ε-neighborhood of L. Another way of saying $x_n \to L$ is: for every ε-neighborhood $V_\varepsilon(L)$ of L there is a point in the sequence after which all elements of (x_n) lie in $V_\varepsilon(L)$.

https://www.desmos.com/calculator/xqtz5biacc

Example Let $x_n = \frac{1}{\sqrt{n}}$ for all $n \in \mathbb{N}$. Prove $\lim_{n \to \infty} x_n = 0$.

Proof Let $\varepsilon > 0$. Define $N \in \mathbb{N}$ such that $\underline{N > \varepsilon^{-2}}$.

Let $n \geq N$. Then observe that

$$\begin{aligned}
|x_n - 0| &= \left| \frac{1}{\sqrt{n}} \right| \\
&= \frac{1}{\sqrt{n}} \\
&\leq \frac{1}{\sqrt{N}} \\
&< \frac{1}{\sqrt{\varepsilon^{-2}}} \\
&= \varepsilon .
\end{aligned}$$

Example Let $x_n = \frac{n}{n+2}$ for all $n \in \mathbb{N}$. Prove $\lim_{n \to \infty} x_n = \underline{1}$.

Proof Let $\varepsilon > 0$. Define $N \in \mathbb{N}$ such that $\underline{N > \frac{2}{\varepsilon}}$.

Let $n \geq N$. Then observe that

$$\begin{aligned}
|x_n - 1| &= \left| \frac{n}{n+2} - 1 \right| \\
&= \left| \frac{n - (n+2)}{n+2} \right| \\
&= \left| \frac{-2}{n+2} \right| \\
&= \frac{2}{n+2}
\end{aligned}$$

$$< \frac{2}{n}$$

$$\leq \frac{2}{N}$$

$$< \varepsilon.$$

Problem 1. Each of the following sequences converges. Make a conjecture about the limit and prove your conjecture.

a. $x_n = 1/n^{1/3}$ for all $n \in \mathbb{N}$

b. $x_n = 2 + (-1)^n/n^2$ for all $n \in \mathbb{N}$

c. $x_n = \cos(n)/(n^3 + 1)$ for all $n \in \mathbb{N}$

(a) We claim $x_n \to 0$. Let $\varepsilon > 0$.

Define $N \in \mathbb{N}$ such that $\underline{N > \varepsilon^{-3}}$.

Let $n \geq N$. Then observe that

$$|x_n - 0| = \left| \frac{1}{n^{1/3}} \right|$$

$$= \frac{1}{n^{1/3}}$$

$$\leq \frac{1}{N^{1/3}}$$

$$< \frac{1}{(\varepsilon^{-3})^{1/3}}$$

$$= \varepsilon.$$

(b) We claim $x_n \to 2$. Let $\varepsilon > 0$.

Define $N \in \mathbb{N}$ such that $\underline{N > \varepsilon^{-1/2}}$.

Let $n \geq N$. Then observe that

$$\begin{aligned} |x_n - 2| &= \left| 2 + \frac{(-1)^n}{n^2} - 2 \right| \\ &= \frac{1}{n^2} \\ &\leq \frac{1}{N^2} \\ &< \frac{1}{(\varepsilon^{-1/2})^2} \\ &= \varepsilon. \end{aligned}$$

(c) We claim $x_n \to 0$. Let $\varepsilon > 0$.

Define $N \in \mathbb{N}$ such that $\underline{N > \varepsilon^{-1/3}}$.

Let $n \geq N$. Then observe that

$$\begin{aligned} |x_n - 0| &= \left| \frac{\cos(n)}{n^3 + 1} \right| \\ &\leq \frac{1}{n^3 + 1} \\ &< \frac{1}{n^3} \\ &\leq \frac{1}{N^3} \\ &< \frac{1}{(\varepsilon^{-1/3})^3} = \varepsilon. \end{aligned}$$

Problem 2. We say that (x_n) *converges* if (x_n) converges to L for some $L \in \mathbb{R}$ and we say (x_n) diverges if it does not converge. Prove that the sequence (x_n) where $x_n = (-1)^n$ for all $n \in \mathbb{N}$ diverges. *Hint: I suggest a proof by contradiction.*

Proof Suppose (x_n) converges. Then there exists $L \in \mathbb{R}$ such that for any $\varepsilon > 0$, there exists $N \in \mathbb{R}$ such that $|x_n - L| < \varepsilon$ for all $n > N$. Therefore if $\varepsilon = 1/2$ there exists $N \in \mathbb{N}$ such that $|x_n - L| < \frac{1}{2}$ whenever $n \geq N$ Suppose $n \geq N$. Then

$$
\begin{aligned}
2 &= |x_n - x_{n+1}| \\
&= |x_n - L + L - x_{n+1}| \\
&\leq |x_n - L| + |x_{n+1} - L| \\
&< 1/2 + 1/2 \\
&= 1
\end{aligned}
$$

which is a contradiction.