§ 2.3 Limit Theorems.

Theorem If a sequence converges, then its limit is unique.

In other words if (x_n) is a sequence such that

$\lim_{n\to\infty} x_n = L_1$ and $\lim_{n\to\infty} x_n = L_2$, then $L_1 = L_2$.

Lemma Let $a, b \in \mathbb{R}$. Then $a = b$ if and only if

$|a-b| < \varepsilon$ for all $\varepsilon > 0$.

Proof of Theorem To show $L_1 = L_2$, it suffices to show

that $|L_1 - L_2| < \varepsilon$ for all $\varepsilon > 0$. Let $\varepsilon > 0$.

Since $x_n \to L_1$, there exists $N_1 \in \mathbb{N}$ such that $|x_n - L_1| < \frac{\varepsilon}{2}$

for all $n \geq N_1$. Since $x_n \to L_2$, there exists $N_2 \in \mathbb{N}$

such that $|x_n - L_2| < \frac{\varepsilon}{2}$ for all $n \geq N_2$. Suppose

$n > \max\{N_1, N_2\}$. Then ...

complete the proof as an exercise.

Def A sequence (x_n) is bounded if there exists $M > 0$ such that $|x_n| \leq M$ for all $n \in \mathbb{N}$.

Theorem If (x_n) converges, then it is bounded.

Proof We must show there exists $M > 0$ such that $|x_n| \leq M$ for all n. Let $L = \lim_{n\to\infty} x_n$. There exists $N \in \mathbb{N}$ such that $|x_n - L| < 1$ for all $n \geq N$.

Therefore
$$\begin{aligned} |x_n| &= |x_n - L + L| \\ &\leq |x_n - L| + |L| \\ &< 1 + |L| \end{aligned}$$

for all $n \geq N$. Let $M = \max\{|x_1|, |x_2|, \ldots, |x_N|, |L|+1\}$.

Then $|x_n| \leq M$ for all $n \in \mathbb{N}$.

Theorem (Algebraic Limit Theorem) Suppose (a_n) and (b_n) converge and $a = \lim_{n\to\infty} a_n$ and $b = \lim_{n\to\infty} b_n$. Then

(1) $\lim_{n\to\infty} (ca_n) = ca$ for any $c \in \mathbb{R}$

(2) $\lim_{n\to\infty} (a_n + b_n) = a + b$

(3) $\lim_{n\to\infty} (a_n b_n) = ab$

(4) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}$ as long as $b \neq 0$ and $b_n \neq 0$ for all $n \in \mathbb{N}$

Proof of (2) To show $a_n + b_n \to a + b$, we must show that for each $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|a_n + b_n - (a+b)| < \varepsilon$ when $n \geq N$. Since $a_n \to a$, there exists $N_1 \in \mathbb{N}$ such that $|a_n - a| < \varepsilon/2$ for all $n \geq N_1$. Since $b_n \to b$, there exists $N_2 \in \mathbb{N}$ such that $|b_n - b| < \varepsilon/2$ for all $n \geq N_2$. Let $N = \max\{N_1, N_2\}$. Suppose $n \geq N$. Observe that

$$
\begin{aligned}
|(a_n + b_n) - (a+b)| &= |(a_n - a) + (b_n - b)| \\
&\leq |a_n - a| + |b_n - b| \\
&< \varepsilon/2 + \varepsilon/2 \\
&= \varepsilon.
\end{aligned}
$$