§2.4 Monotone Convergence Theorem

Def A sequence (a_n) is <u>increasing</u> if $a_n \le a_{n+1}$ for all $n \in \mathbb{N}$; <u>decreasing</u> if $a_n \ge a_{n+1}$ for all $n \in \mathbb{N}$.

If (a_n) is either increasing or decreasing, it is called <u>monotone</u>.

Theorem (Monotone Convergence Theorem) If a sequence is bounded and monotone, then it converges.

More precisely,

(1) if (a_n) is increasing and bounded above, then it converges.

(2) if (a_n) is decreasing and bounded below, then it converges.

Proof We'll do (1). Decreasing case is an exercise.

Since (a_n) is bounded above we let $s = \sup\{a_n : n \in \mathbb{N}\}$.

We claim $a_n \to s$. Let $\varepsilon > 0$. We must show there exists $N \in \mathbb{N}$ such that $|a_n - s| < \varepsilon$ for all $n \ge N$.

Since $s - \varepsilon < s$, $s - \varepsilon$ is not an upper bound of $\{a_n : n \in \mathbb{N}\}$. Thus there exists $N \in \mathbb{N}$ such that $s - \varepsilon < a_N$. Since (a_n) is increasing $a_N \le a_n$ for all $n \ge N$. Suppose $n \ge N$. Then

$$s - \varepsilon < a_N \le a_n \le s < s + \varepsilon,$$

which implies $|a_n - s| < \varepsilon$.

Example Let $y_1 = 6$ and for each $n \in \mathbb{N}$, define

$$y_{n+1} = \frac{2y_n - 6}{3}.$$ Prove (y_n) converges and find its limit.

In Exercise 1.2.12, we showed $y_n > -6$ for all $n \in \mathbb{N}$ and proved (y_n) is decreasing. So by the monotone convergence theorem, (y_n) converges to some $L \in \mathbb{R}$. Taking the limit as $n \to \infty$ of $\frac{2y_n - 6}{3}$, by the Algebraic Limit Theorem, we get $L = \frac{2L-6}{3}$, which implies $L = -6$.

Def Let (b_n) be a sequence. An *infinite series* is a formal expression of the form

$$\sum_{n=1}^{\infty} b_n = b_1 + b_2 + b_3 + \cdots.$$

The *sequence of partial sums* (s_m) is given by

$$s_m = b_1 + b_2 + \cdots + b_m.$$

We say $\sum_{n=1}^{\infty} b_n$ *converges to* B if $s_m \to B$. If (s_m) diverges, we say $\sum_{n=1}^{\infty} b_n$ diverges.

Examples

1. $\sum_{n=1}^{\infty} 1 = 1 + 1 + 1 + \cdots$

2. $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$

3. $\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \cdots$