Math 301 — Infinite series

Problem 1. For each of the following series, explain whether it converges or diverges. Recall that we have proved $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. You should take these facts as given.

a. $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$

b. $\sum_{n=2}^{\infty} \frac{n^2}{n^3-1}$

c. $\sum_{n=2}^{\infty} \frac{1}{\ln n}$

d. $\sum_{n=1}^{\infty} \frac{n}{n+1}$

Problem 2. For each of the following statements, determine whether it is true or false and explain why.

a. If (a_n) is a Cauchy sequence, then the series $\sum_{k=1}^{\infty} a_k$ converges.

b. If the series $\sum_{k=1}^{\infty} a_k$ converges, then (a_n) is a Cauchy sequence.

c. If $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ both diverge, then $\sum_{k=1}^{\infty} a_k b_k$ diverges.

Problem 3. Suppose $r \in \mathbb{R}$ is a given constant. Notice that when $r \neq 1$, for any $m \in \mathbb{N}$

$$(1-r)(1+r+r^2+\cdots+r^{m-1}) = 1-r^m.$$

a. Use this fact to write a formula for the partial sum s_m of the series $\sum_{k=0}^{\infty} r^k$.

b. Explain why the series $\sum_{k=0}^{\infty} r^k$ converges if and only if $|r| < 1$. You may take as given that $r^m \to 0$ if and only if $|r| < 1$.

c. When $|r| < 1$ what does the series $\sum_{k=0}^{\infty} r^k$ converge to?