Math 301 — Exam 1 review guide

Your first exam will have two parts, an in-class portion on March 7 and a take-home portion due March 10. The in-class portion will contain about 5 problems, some with multiple parts. You should expect to see a question asking you to state some definitions or theorems, a question asking you to prove a statement that was proved in class, and a few questions in the style similar to worksheets, homework, and quizzes. You should expect to write 2-3 proofs. It will cover material from Homework 0 to Homework 4. In the textbook, this is material spanning Sections 1.1-2.5, excluding sections 1.6 and 1.7. I have outlined some important definitions, theorems, and general topics below. Also, the problems below give you a sampling of some problems like those that will appear on the exam, but it's not necessarily comprehensive, so make sure to review old homework, quizzes, worksheets, and lecture notes.

Definitions and theorems

While not necessarily comprehensive, here is an absolutely-must-know list of definitions.

- upper bound, lower bound, maximum, minimum, supremum, infimum of a set
- countable and uncountable sets
- sequence, convergent sequence, divergent sequence, monotone sequence, bounded sequence, subsequence

Similarly, here is an absolutely-must-know list of statements you should know.

- the Completeness Axiom, the Archimedean Property, the density of $\mathbb{Q}$ in $\mathbb{R}$, the Nested Interval Property
- the Algebraic Limit Theorem, the Order Limit Theorem, Monotone Convergence Theorem, Bolzano-Weierstrass Theorem
- Every convergent sequence is bounded and has a unique limit and every subsequence converges to that limit.

Proofs and logic

You should know the following about proofs and logic. You should know how to:

- Negate statements involving quantifiers.
- Prove recursively defined sequences are bounded or monotone using induction.
- Prove a certain value is the supremum or infimum of a set.
- Prove two given sets have the same cardinality.
- Prove a given sequence converges or diverges.

Sample problems

These problems are not comprehensive and there are more than you will see on the exam itself, but will give you an idea of the kinds of questions to expect.

Problem 1. Please state whether the following statements are true or false. If a statement is true, explain why. If a statement is false, give a counterexample and modify the statement slightly to make it true.

a. Every subset of $\mathbb{R}$ that is nonempty and bounded above has a unique upper bound.

b. The interval $(-\pi/2, \pi/2)$ has the same cardinality as the interval $(0, \infty)$.

c. If $a_n \to a$ and $b_n \to b$ and $a_n < b_n$ for all $n \in \mathbb{N}$ then $a < b$.

d. If (a_n) converges and (b_n) is bounded, then $(a_n b_n)$ converges.

e. If (a_n) is bounded and contains a convergent subsequence, then (a_n) converges.

Problem 2. Prove the following statements:

a. Let $S \subseteq \mathbb{R}$ be a non-empty set that is bounded below and let $-S = \{-x : x \in S\}$. Prove that $-S$ is bounded above and $\sup(-S) = -\inf S$.

b. Prove that $\sup\left\{1 - \frac{1}{n^2} : n \in \mathbb{N}\right\} = 1$ using the Archimedean Property.

c. Let $p > 0$. Prove that the sequence (a_n), where $a_n = 1/n^p$ for all $n \in \mathbb{N}$, converges.

d. Let $a_1 = 3$ and $a_{n+1} = \frac{1}{4 - a_n}$ for all $n \in \mathbb{N}$. Prove that (a_n) converges and find its limit.

e. Prove that if (a_n) diverges to $+\infty$ and $k > 0$ then (ka_n) diverges to $+\infty$ as well.