Math 301 — More on series

Summary. As we go through and prove the series tests that arise in second semester calculus, it's also useful to go back and review how they're used to prove that certain examples converge or diverge. The following question asks you to use a variety of the tests we've gone through so far. Then try proving the ratio test in the case when $L > 1$.

Problem 1. For each of the following series, decide whether it converges.

a. $\sum (-1)^n \frac{2^n}{n!}$

b. $\sum (-1)^n \frac{3^{2n}}{8^n n^3}$

c. $\sum (-1)^n \frac{1}{5n-3}$

Problem 2. Prove the ratio test in the case when $L > 1$ by emulating the proof of the $L < 1$ case.

Problem 1

(a) Let $L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$

$$= \lim_{n\to\infty} \frac{2^{n+1}}{(n+1)!} \frac{n!}{2^n}$$

$$= \lim_{n\to\infty} \frac{2}{n+1} = 0.$$

Then $\sum a_n$ converges by ratio test

since $L < 1$.

(b) Let $L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$

$$= \lim_{n\to\infty} \frac{9^{n+1}}{8^{n+1}(n+1)^3} \cdot \frac{8^n (n+1)^3}{9^n}$$

$$= \frac{9}{8}.$$

Then $\sum a_n$ diverges by the ratio test since $L > 1$.

(c) We didn't discuss this together, but the alternating series test must be used.

Let $b_n = \frac{1}{5n-3}$. Then $b_n \geq 0$ for all n, b_n is decreasing since

$$b_{n+1} = \frac{1}{5(n+1)-3} \leq \frac{1}{5n-3} = b_n,$$

and $\lim b_n = \lim \frac{1}{5n+3} = 0$. So

the series converges by alt. series test.

Problem 2 this is homework.