Math 301 — Suprema and infima

Summary. Consider a non-empty set $S \subseteq \mathbb{R}$. An *upper bound* for S is a number M such that $a \leq M$ for all $a \in S$. The set S is *bounded above* if it has an upper bound $M < \infty$. The *supremum* of S, written as $\sup S$, is the least (or smallest) upper bound. That is, $a \leq \sup S$ for all $a \in S$ and $\sup S \leq M$ for any upper bound M. The *maximum* of S is a number $M \in S$ such that $a \leq M$ for all $a \in S$. The definition of $\max S$ is subtle. Notice that M must be an element of S; that's the important part! There are similar definitions for *lower bound, bounded below, infimum,* and *minimum.*

Problem 1. For each set below, please give

- Two upper bounds and two lower bounds,
- the supremum and infimum,
- the maximum and minimum, if either exists.

a. The closed interval $[-2, 4] = \{x \in \mathbb{R} : -2 \leq x \leq 4\}$.

b. The open interval $(-2, 4) = \{x \in \mathbb{R} : -2 < x < 4\}$.

c. The half open interval $(-2, 4] = \{x \in \mathbb{R} : -2 < x \leq 4\}$.

d. $\left\{r \in \mathbb{Q} : r^2 < 2\right\}$

e. $\left\{r \in \mathbb{Q} : r^2 \leq 9\right\}$

f. $\{1/n : n \in \mathbb{N}\}$

g. $\bigcap_{n=1}^{\infty}(-1/n, 1 + 1/n)$

Problem 2. When does $\max S$ exist? When $\max S$ exists, how is it related to $\sup S$?