§ 1.3 The Axiom of Completeness

Definition A nonempty set $A \subseteq \mathbb{R}$ is said to be

① bounded above if there exists $M \in \mathbb{R}$ such that $x \leq M$ for all $x \in A$. We call M an upper bound of A.

② bounded below if there exists $m \in \mathbb{R}$ such that $x \geq m$ for all $x \in A$. We call m a lower bound of A.

③ bounded if it is bounded above and bounded below.

Examples Decide whether each of the following is bounded above, bounded below, or bounded. For those bounded above give 2 examples of upper bounds. Similarly give 2 examples of lower bounds for those that are bounded above.

① $A = \{ x \in \mathbb{R} : x^2 \leq 5 \} = [-\sqrt{5}, \sqrt{5}]$

② $B = \{ x \in \mathbb{R} : x^3 < 5 \} = (-\infty, 5^{1/3})$

③ $C = \{ x \in \mathbb{N} : x \leq 5 \} = \{1, 2, 3, 4, 5\}$

④ $D = \{ x \in \mathbb{Q} : x^2 \leq 2 \} = [-\sqrt{2}, \sqrt{2}] \cap \mathbb{Q}$

⑤ $E = \{ x \in \mathbb{Q} : x^2 < 2 \} = (-\sqrt{2}, \sqrt{2}) \cap \mathbb{Q}$

Definition A real number M is a *maximum* of a set $A \subseteq \mathbb{R}$ if $x \le M$ for all $x \in A$ *and* $M \in A$. A real number m is a *minimum* of A if $x \ge m$ for all $x \in A$ *and* $m \in A$.

Example Which sets in the previous example have a maximum? A minimum? If the max/min exists, what is it?

① $\max A = \sqrt{5}$, $\min A = -\sqrt{5}$

③ $\max C = 5$, $\min C = 1$

②, ④, ⑤ no max, no min

Definition Let $A \subseteq \mathbb{R}$ be a nonempty set that is bounded above. Then $U \in \mathbb{R}$ is said to be the *supremum* or *least upper bound* of A if

① $x \le U$ for all $x \in A$

② If M is an upper bound of A, then $U \le M$.

We denote the supremum of A by $\sup A$. If A is not bounded above, we let $\sup A = +\infty$.

Examples Which sets in the previous examples have a supremum? If it exists, what is it?

① $\sup A = \sqrt{5}$, ② $\sup B = 5^{1/3}$. ③ $\sup C = 5$, ④, ⑤ $\sup D = \sup E = \sqrt{2}$

Distinguishing $\mathbb{Q}$ from $\mathbb{R}$.

Notice the set $\{ r \in \mathbb{Q} : r^2 < 2 \}$ is non-empty and bounded above and has a supremum ($\sqrt{2}$).

However its supremum is not a rational number; we need more than just rationals to define the supremum of the set despite the fact that it only contains rationals. In this sense $\mathbb{Q}$ is "incomplete."

Axiom of Completeness Every nonempty set of real numbers that is bounded above has a least upper bound.

"Informal" Definition of $\mathbb{R}$ The real numbers consist of $\mathbb{Q}$ along with suprema of nonempty sets of rationals that are bounded above.

"Formal" Definition of $\mathbb{R}$ It is an ordered field (number system with addition, mult., inverses, $<$) that contains $\mathbb{Q}$ and satisfies the Axiom of Completeness.

Example Let $A \subseteq \mathbb{R}$ be a non-empty set that is bounded above and let $c \in \mathbb{R}$. Define the set B by

$$B = \{c + a : a \in A\}.$$

Prove that $\sup B = c + \sup A$.

We will first show that $c + \sup A$ is an upper bound of B. Let $b \in B$. Then $b = c + a$ for some $a \in A$. Since $a \leq \sup A$, $b = c + a \leq c + \sup A$. Since b was arbitrary, $c + \sup A$ is an upper bound of B.

Next, we will show $c + \sup A$ is the least upper bound of B. Let U be an arbitrary upper bound of B. We must show $c + \sup A \leq U$. To show this, we will show $\sup A \leq U - c$ by showing $U - c$ is an upper bound of A. Let $a \in A$. Then $c + a \in B$ and so $c + a \leq U$. Therefore $a \leq U - c$, which shows $U - c$ is an upper bound of A. Since $\sup A$ is the least upper bound of A, $\sup A \leq U - c$.

Lemma Let $S \subseteq \mathbb{R}$ be a set that is bounded above and let $\alpha \in \mathbb{R}$ be an upper bound for S. Then $\alpha = \sup S$ if and only if for every $\varepsilon > 0$, there exists $x \in S$ such that $x > \alpha - \varepsilon$.

Proof ($\Longrightarrow$) Assume $\alpha = \sup S$. Let $\varepsilon > 0$. Then $\alpha - \varepsilon < \alpha$, so $\alpha - \varepsilon$ is not an upper bound for S (since α is the least upper bound). Therefore there exists $x \in S$ such that $x > \alpha - \varepsilon$.

($\Longleftarrow$) Assume $\forall \varepsilon > 0 \ \exists \ x \in S$ such that $x > \alpha - \varepsilon$. Let u be an upper bound for S. We must prove $\alpha \leq u$. Suppose $u < \alpha$ and let $\varepsilon_0 = \alpha - u$. Then there exists $x \in S$ such that

$$x > \alpha - \varepsilon_0 = \alpha - (\alpha - u) = u$$

but this contradicts that u is an upper bound.

Problem 1. Let S and T be non-empty, bounded subsets of $\mathbb{R}$ and suppose that $S \subseteq T$. Prove that $\sup S \leq \sup T$.

We'll prove that $\sup T$ is an upper bound of S. This will imply $\sup S \leq \sup T$ since $\sup S$ is the <u>least</u> upper bound of S.

Let $x \in S$. Then $x \in T$. This implies $x \leq \sup T$ since $\sup T$ is an upper bound of T. Therefore $\sup T$ is an upper bound of S.

Problem 2. Let $A, B \subseteq \mathbb{R}$ be given sets that are nonempty and bounded above. Define $C = \{a + b : a \in A, b \in B\}$. Use today's lemma to prove that $\sup C = \sup A + \sup B$.

Let $\varepsilon > 0$. We must show there exists $c \in C$ such that $c > \sup A + \sup B - \varepsilon$. Observe that by the Lemma, there exists $a \in A$ such that $a > \sup A - \frac{\varepsilon}{2}$. Similarly, there exists $b \in B$ such that $b > \sup B - \frac{\varepsilon}{2}$. Let $c = a + b$. Then $c \in C$ and

$$c = a + b > \sup A - \varepsilon/2 + \sup B - \varepsilon/2$$
$$= \sup A + \sup B - \varepsilon.$$

Problem 3. For each set below, give its maximum and minimum (if they exist), as well as its supremum and infimum.

a. The half open interval $(-2, 4] = \{x \in \mathbb{R} : -2 < x \leq 4\}$.

b. $\{1/n : n \in \mathbb{N}\}$

c. $\bigcap_{n=1}^{\infty}(-1/n, 1 + 1/n)$

d. $\{r \in \mathbb{Q} : 0 < r^2 \leq 2\}$

e. $\{r \in \mathbb{R} : 0 < r^2 \leq 2\}$

ⓐ min DNE, inf $= -2$, max $=$ sup $= 4$

ⓑ min DNE, inf $= 0$, max $=$ sup $= 1$

ⓒ set is $[0, 1]$, so min $=$ inf $= 0$, max $=$ sup $= 1$

ⓓ set is $\mathbb{Q} \cap [-\sqrt{2}, \sqrt{2}] \setminus \{0\}$

min DNE, inf $= -\sqrt{2}$, max DNE, sup $= \sqrt{2}$

ⓔ $[-\sqrt{2}, \sqrt{2}] \setminus \{0\}$ min $=$ inf $= -\sqrt{2}$, max $=$ sup $= \sqrt{2}$